

PRIMAL AND SHADOW FUNCTIONS, DUAL AND DUAL-SHADOW FUNCTIONS FOR A CIRCULAR CRACK AND A CIRCULAR 90° V-NOTCH WITH NEUMANN BOUNDARY CONDITIONS

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ABSTRACT. This report presents explicit analytical expressions for the primal, primal shadows, dual and dual shadows functions for the Laplace equation in the vicinity of a circular singular edge with Neumann boundary conditions on the faces that intersect at the singular edge. Two configurations are investigated: a penny-shaped crack and a 90° V-notch.

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INTRODUCTION

The edges that we consider are in an axisymmetric configuration. Cylindrical coordinates are denoted by (r, θ, x_3) with the distance r to the axis, the rotation angle $\theta \in [0, 2\pi]$ and the coordinate x_3 along the axis. The circular edges are generated by rotating a point P in the r - x_3 half-plane around the x_3 axis. Let R be $r(P)$. We consider the polar coordinate system (ρ, φ) centered at P such that the following relations hold

$$r = R + \rho \cos \varphi \quad \text{and} \quad x_3 = \rho \sin \varphi.$$

In the vicinity of the edge the domain coincides with the set

$$\{(\rho, \varphi, \theta), \quad 0 < \rho < \rho_0, \quad \varphi_1 < \varphi < \varphi_2, \quad \theta \in [0, 2\pi]\}.$$

In this report we address the cases of

- the penny-shaped crack, for which $\varphi_1 = -\pi$ and $\varphi_2 = \pi$
- the 90° V-notch that corresponds to $\varphi_1 = -\pi$ and $\varphi_2 = \pi/2$.

Solutions τ of the homogeneous Laplace equation $\Delta\tau = 0$ with zero Neumann boundary conditions on the faces abutting the edge can be expressed as follows [1] in coordinates (ρ, φ, θ) :

$$\tau(\rho, \varphi, \theta) = \sum_{j \geq 0} \sum_{h=0,2,4,\dots} \partial_\theta^h A_j(\theta) \rho^{\alpha_j} \sum_{f \geq 0} \left(\frac{\rho}{R}\right)^{h+f} \phi_{h,j,f}(\varphi), \quad (1)$$

The functions A_j are coefficients which depend on τ , whereas the functions $\phi_{h,j,f}$ are completely determined by the geometry of the domain around the edge, I.E. by the limiting angles φ_1 and φ_2 . For each natural number j , the term $\phi_{0,j,0}$ is an eigenfunction of a 1D Laplace operator and for $h+j > 0$, the “shadows” $\phi_{h,j,f}(\varphi)$ are obtained by solving the following recursive set of equations, cf [1, eq. (15-17) and (36)].

For $h = 0$

$$\alpha_j^2 \phi_{0,j,0} + \phi_{0,j,0}'' = 0 \quad (2)$$

$$(\alpha_j + 1)^2 \phi_{0,j,1} + \phi_{0,j,1}'' = -(\alpha_j \cos \varphi \phi_{0,j,0} - \sin \varphi \phi_{0,j,0}') \quad (3)$$

$$(\alpha_j + i)^2 \phi_{0,j,f} + \phi_{0,j,f}'' = -[(\alpha_j + f)(\alpha_j + f - 1) \cos \varphi \phi_{0,j,f-1} - \sin \varphi \phi_{0,j,f-1}' + \cos \varphi \phi_{0,j,f-1}''] \quad (4)$$

$$f \geq 2$$

For $h = 2, 4, 6, \dots$,

$$\begin{aligned} (\alpha_j + f + h)^2 \phi_{h,j,f} + \phi_{h,j,f}'' &= -(h + f + \alpha_j - 1) [2(h + f + \alpha_j) - 1] \cos \varphi \phi_{h,j,f-1} \\ &\quad + \sin \varphi \phi_{h,j,f-1}' - 2 \cos \varphi \phi_{h,j,f-1}'' \\ &\quad - (h + \alpha_j + f - 2)(h + \alpha_j + f - 1) \cos^2 \varphi \phi_{h,j,f-2} \\ &\quad + \cos \varphi \sin \varphi \phi_{h,j,f-2}' - \cos^2 \varphi \phi_{h,j,f-2}'' - \partial_\varphi \phi_{(h-2),j,f} \end{aligned} \quad (5)$$

These equations have to be completed by the Neumann boundary conditions:

$$\partial_\varphi \phi_{h,j,f} = 0, \quad \text{on } \varphi = \varphi_1 \quad \text{and} \quad \varphi = \varphi_2 \quad (6)$$

Equations (2-4) with $h = 0$ are associated with the axisymmetric case, and for $h = 2, 4, 6, \dots$ equations (5) are associated with the non-axisymmetric case.

Each eigenfunction and shadow, has a dual and dual shadow counterpart, that can be computed by the following recursive system.

For the dual eigenfunctions and their shadows $\psi_{0,j,f}(\varphi)$ (for $h = 0$ associated with the axisymmetric case), the recursive equation to be solved is [1]:

For $h = 0$

$$\alpha_j^2 \psi_{0,j,0} + \psi_{0,j,0}'' = 0, \quad (7)$$

$$(-\alpha_j + 1)^2 \psi_{0,j,1} + \psi_{0,j,1}'' = -(-\alpha_j \cos \varphi \psi_{0,j,0} - \sin \varphi \psi_{0,j,0}') , \quad (8)$$

$$\begin{aligned} (-\alpha_j + i)^2 \psi_{0,j,f} + \psi_{0,j,f}'' &= -[(-\alpha_j + f)(-\alpha_j + f - 1) \cos \varphi \psi_{0,j,f-1} \\ &\quad - \sin \varphi \psi_{0,j,f-1}' + \cos \varphi \psi_{0,j,f-1}''] , \quad f \geq 2, \end{aligned} \quad (9)$$

For $h = 2, 4, 6, \dots$,

$$\begin{aligned} (-\alpha_j + f + h)^2 \psi_{h,j,f} + \psi_{h,j,f}'' &= \\ &\quad -(h + f - \alpha_j - 1) [2(h + f - \alpha_j) - 1] \cos \varphi \psi_{h,j,f-1} + \sin \varphi \psi_{h,j,f-1}' \\ &\quad - 2 \cos \varphi \psi_{h,j,f-1}'' - (h - \alpha_j + f - 2)(h - \alpha_j + f - 1) \cos^2 \varphi \psi_{h,j,f-2} \\ &\quad + \cos \varphi \sin \varphi \psi_{h,j,f-2}' - \cos^2 \varphi \psi_{h,j,f-2}'' - \psi_{(h-2),j,f} \end{aligned} \quad (10)$$

with homogeneous Neumann boundary conditions

$$\partial_\varphi \psi_{h,j,f} = 0 \quad \text{on } \varphi = \varphi_1 \quad \text{and} \quad \varphi = \varphi_2. \quad (11)$$

The above systems of equations have been solved using the formal calculus software Mathematica.

REFERENCES

- [1] Z. Yosibash, S. Shannon, M. Dauge, and M. Costabel. Circular edge singularities for the Laplace equation and the elasticity system in 3-D domains. *Int. Jour. Fracture*, 168:31–52, 2011. hal.archives-ouvertes.fr/hal-00490971

1. PRIMAL FUNCTIONS AND SHADOWS FOR THE CRACK $-\pi \leq \varphi \leq \pi$ 1.1. First singular exponent ($j = 1$).

$$\phi_{0,1,0} = \sin \frac{\varphi}{2}$$

$$\phi_{0,1,1} = \frac{1}{4} \sin \frac{\varphi}{2}$$

$$\phi_{0,1,2} = \frac{1}{12} \sin \frac{\varphi}{2} - \frac{3}{32} \sin \frac{3\varphi}{2}$$

$$\phi_{0,1,3} = \frac{1}{16} \sin \frac{\varphi}{2} - \frac{1}{30} \sin \frac{3\varphi}{2} + \frac{5}{128} \sin \frac{5\varphi}{2}$$

$$\phi_{0,1,4} = \frac{3}{80} \sin \frac{\varphi}{2} - \frac{5}{128} \sin \frac{3\varphi}{2} + \frac{1}{70} \sin \frac{5\varphi}{2} - \frac{35}{2048} \sin \frac{7\varphi}{2}$$

$$\phi_{0,1,5} = \frac{1}{32} \sin \frac{\varphi}{2} - \frac{3}{140} \sin \frac{3\varphi}{2} + \frac{35}{1536} \sin \frac{5\varphi}{2} - \frac{2}{315} \sin \frac{7\varphi}{2} + \frac{63}{8192} \sin \frac{9\varphi}{2}$$

$$\phi_{0,1,6} = \frac{5}{224} \sin \frac{\varphi}{2} - \frac{35}{1536} \sin \frac{3\varphi}{2} + \frac{1}{84} \sin \frac{5\varphi}{2} - \frac{105}{8192} \sin \frac{7\varphi}{2} + \frac{2}{693} \sin \frac{9\varphi}{2} - \frac{231}{65536} \sin \frac{11\varphi}{2}$$

$$\begin{aligned} \phi_{0,1,7} = & \frac{5}{256} \sin \frac{\varphi}{2} - \frac{5}{336} \sin \frac{3\varphi}{2} + \frac{63}{4096} \sin \frac{5\varphi}{2} - \frac{1}{154} \sin \frac{7\varphi}{2} + \frac{231}{32768} \sin \frac{9\varphi}{2} - \frac{4}{3003} \sin \frac{11\varphi}{2} \\ & + \frac{429}{262144} \sin \frac{13\varphi}{2} \end{aligned}$$

$$\begin{aligned} \phi_{0,1,8} = & \frac{35}{2304} \sin \frac{\varphi}{2} - \frac{63}{4096} \sin \frac{3\varphi}{2} + \frac{5}{528} \sin \frac{5\varphi}{2} - \frac{1617}{163840} \sin \frac{7\varphi}{2} + \frac{1}{286} \sin \frac{9\varphi}{2} - \frac{1001}{262144} \sin \frac{11\varphi}{2} \\ & + \frac{4}{6435} \sin \frac{13\varphi}{2} - \frac{6435}{8388608} \sin \frac{15\varphi}{2} \end{aligned}$$

$$\begin{aligned} \phi_{0,1,9} = & \frac{7}{512} \sin \frac{\varphi}{2} - \frac{35}{3168} \sin \frac{3\varphi}{2} + \frac{231}{20480} \sin \frac{5\varphi}{2} - \frac{5}{858} \sin \frac{7\varphi}{2} + \frac{1001}{163840} \sin \frac{9\varphi}{2} \\ & - \frac{4}{2145} \sin \frac{11\varphi}{2} + \frac{2145}{1048576} \sin \frac{13\varphi}{2} - \frac{32}{109395} \sin \frac{15\varphi}{2} + \frac{12155}{33554432} \sin \frac{17\varphi}{2} \end{aligned}$$

$$\begin{aligned} \phi_{0,1,10} = & \frac{63}{5632} \sin \frac{\varphi}{2} - \frac{231}{20480} \sin \frac{3\varphi}{2} + \frac{35}{4576} \sin \frac{5\varphi}{2} - \frac{1287}{163840} \sin \frac{7\varphi}{2} + \frac{1}{286} \sin \frac{9\varphi}{2} \\ & - \frac{3861}{1048576} \sin \frac{11\varphi}{2} + \frac{12}{12155} \sin \frac{13\varphi}{2} - \frac{36465}{33554432} \sin \frac{15\varphi}{2} + \frac{32}{230945} \sin \frac{17\varphi}{2} \\ & - \frac{46189}{268435456} \sin \frac{19\varphi}{2} \end{aligned}$$

$$\begin{aligned}
\phi_{2,1,0} &= -\frac{1}{6} \sin \frac{\varphi}{2} \\
\phi_{2,1,1} &= -\frac{1}{8} \sin \frac{\varphi}{2} + \frac{7}{60} \sin \frac{3\varphi}{2} \\
\phi_{2,1,2} &= -\frac{31}{240} \sin \frac{\varphi}{2} + \frac{5}{64} \sin \frac{3\varphi}{2} - \frac{19}{280} \sin \frac{5\varphi}{2} \\
\phi_{2,1,3} &= -\frac{41}{384} \sin \frac{\varphi}{2} + \frac{111}{1120} \sin \frac{3\varphi}{2} - \frac{35}{768} \sin \frac{5\varphi}{2} + \frac{187}{5040} \sin \frac{7\varphi}{2} \\
\phi_{2,1,4} &= -\frac{8209}{80640} \sin \frac{\varphi}{2} + \frac{119}{1536} \sin \frac{3\varphi}{2} - \frac{271}{4032} \sin \frac{5\varphi}{2} + \frac{105}{4096} \sin \frac{7\varphi}{2} - \frac{437}{22176} \sin \frac{9\varphi}{2} \\
\phi_{2,1,5} &= -\frac{4069}{46080} \sin \frac{\varphi}{2} + \frac{39899}{483840} \sin \frac{3\varphi}{2} - \frac{427}{8192} \sin \frac{5\varphi}{2} + \frac{1259}{29568} \sin \frac{7\varphi}{2} - \frac{231}{16384} \sin \frac{9\varphi}{2} \\
&\quad + \frac{1979}{192192} \sin \frac{11\varphi}{2} \\
\phi_{2,1,6} &= -\frac{483047}{5806080} \sin \frac{\varphi}{2} + \frac{5663}{81920} \sin \frac{3\varphi}{2} - \frac{18467}{304128} \sin \frac{5\varphi}{2} + \frac{5467}{163840} \sin \frac{7\varphi}{2} - \frac{2837}{109824} \sin \frac{9\varphi}{2} \\
&\quad + \frac{1001}{131072} \sin \frac{11\varphi}{2} - \frac{4387}{823680} \sin \frac{13\varphi}{2} \\
\phi_{2,1,7} &= -\frac{153325}{2064384} \sin \frac{\varphi}{2} + \frac{8920073}{127733760} \sin \frac{3\varphi}{2} - \frac{49621}{983040} \sin \frac{5\varphi}{2} + \frac{331871}{7907328} \sin \frac{7\varphi}{2} \\
&\quad - \frac{27027}{1310720} \sin \frac{9\varphi}{2} + \frac{50111}{3294720} \sin \frac{11\varphi}{2} - \frac{2145}{524288} \sin \frac{13\varphi}{2} + \frac{76627}{28005120} \sin \frac{15\varphi}{2} \\
\phi_{2,1,8} &= -\frac{79481719}{1135411200} \sin \frac{\varphi}{2} + \frac{2514853}{41287680} \sin \frac{3\varphi}{2} - \frac{6664667}{123002880} \sin \frac{5\varphi}{2} + \frac{275561}{7864320} \sin \frac{7\varphi}{2} \\
&\quad - \frac{81201}{2928640} \sin \frac{9\varphi}{2} + \frac{13013}{1048576} \sin \frac{11\varphi}{2} - \frac{327121}{37340160} \sin \frac{13\varphi}{2} + \frac{36465}{16777216} \sin \frac{15\varphi}{2} \\
&\quad - \frac{165409}{118243840} \sin \frac{17\varphi}{2} \\
\\
\phi_{4,1,0} &= \frac{1}{120} \sin \frac{\varphi}{2} \\
\phi_{4,1,1} &= \frac{1}{96} \sin \frac{\varphi}{2} - \frac{3}{280} \sin \frac{3\varphi}{2} \\
\phi_{4,1,2} &= \frac{17}{1008} \sin \frac{\varphi}{2} - \frac{7}{768} \sin \frac{3\varphi}{2} + \frac{19}{2016} \sin \frac{5\varphi}{2} \\
\phi_{4,1,3} &= \frac{41}{2304} \sin \frac{\varphi}{2} - \frac{433}{24192} \sin \frac{3\varphi}{2} + \frac{7}{1024} \sin \frac{5\varphi}{2} - \frac{13}{1848} \sin \frac{7\varphi}{2} \\
\phi_{4,1,4} &= \frac{1019}{46080} \sin \frac{\varphi}{2} - \frac{1}{64} \sin \frac{3\varphi}{2} + \frac{595}{38016} \sin \frac{5\varphi}{2} - \frac{77}{16384} \sin \frac{7\varphi}{2} + \frac{263}{54912} \sin \frac{9\varphi}{2} \\
\phi_{4,1,5} &= \frac{1367 \sin \frac{\varphi}{2}}{61440} - \frac{5617 \sin \frac{3\varphi}{2}}{253440} + \frac{605 \sin \frac{5\varphi}{2}}{49152} - \frac{24155 \sin \frac{7\varphi}{2}}{1976832} + \frac{1001 \sin \frac{9\varphi}{2}}{327680} - \frac{2533 \sin \frac{11\varphi}{2}}{823680} \\
\phi_{4,1,6} &= \frac{3211069 \sin \frac{\varphi}{2}}{127733760} - \frac{58267 \sin \frac{3\varphi}{2}}{2949120} + \frac{514259 \sin \frac{5\varphi}{2}}{26357760} - \frac{4433 \sin \frac{7\varphi}{2}}{491520} + \frac{81 \sin \frac{9\varphi}{2}}{9152} \\
&\quad - \frac{1001 \sin \frac{11\varphi}{2}}{524288} + \frac{35401 \sin \frac{13\varphi}{2}}{18670080}
\end{aligned}$$

$$\begin{aligned}
\phi_{6,1,0} &= -\frac{1}{5040} \sin \frac{\varphi}{2} \\
\phi_{6,1,1} &= -\frac{1}{2880} \sin \frac{\varphi}{2} + \frac{11}{30240} \sin \frac{3\varphi}{2} \\
\phi_{6,1,2} &= -\frac{13}{17280} \sin \frac{\varphi}{2} + \frac{1}{2560} \sin \frac{3\varphi}{2} - \frac{1}{2376} \sin \frac{5\varphi}{2} \\
\phi_{6,1,3} &= -\frac{1}{1024} \sin \frac{\varphi}{2} + \frac{191}{190080} \sin \frac{3\varphi}{2} - \frac{11}{30720} \sin \frac{5\varphi}{2} + \frac{97}{247104} \sin \frac{7\varphi}{2} \\
\phi_{6,1,4} &= -\frac{511}{345600} \sin \frac{\varphi}{2} + \frac{187}{184320} \sin \frac{3\varphi}{2} - \frac{1751}{1647360} \sin \frac{5\varphi}{2} + \frac{143}{491520} \sin \frac{7\varphi}{2} - \frac{59}{183040} \sin \frac{9\varphi}{2}
\end{aligned}$$

$$\begin{aligned}
\phi_{8,1,0} &= \frac{1}{362880} \sin \frac{\varphi}{2} \\
\phi_{8,1,1} &= \frac{\sin \frac{\varphi}{2}}{161280} - \frac{13 \sin \frac{3\varphi}{2}}{1995840} \\
\phi_{8,1,2} &= \frac{89 \sin \frac{\varphi}{2}}{5322240} - \frac{11 \sin \frac{3\varphi}{2}}{1290240} + \frac{107 \sin \frac{5\varphi}{2}}{11531520}
\end{aligned}$$

$$\phi_{10,1,0} = -\frac{1}{39916800} \sin \frac{\varphi}{2}$$

1.2. Second singular exponent ($j = 3$).

$$\begin{aligned}
\phi_{0,3,0} &= \sin \frac{3\varphi}{2} \\
\phi_{0,3,1} &= -\frac{1}{4} \sin \frac{\varphi}{2} \\
\phi_{0,3,2} &= -\frac{3}{32} \sin \frac{\varphi}{2} + \frac{1}{10} \sin \frac{3\varphi}{2} \\
\phi_{0,3,3} &= -\frac{3}{40} \sin \frac{\varphi}{2} + \frac{5}{128} \sin \frac{3\varphi}{2} - \frac{3}{70} \sin \frac{5\varphi}{2} \\
\phi_{0,3,4} &= -\frac{3}{64} \sin \frac{\varphi}{2} + \frac{27}{560} \sin \frac{3\varphi}{2} - \frac{35}{2048} \sin \frac{5\varphi}{2} + \frac{2}{105} \sin \frac{7\varphi}{2} \\
\phi_{0,3,5} &= -\frac{9}{224} \sin \frac{\varphi}{2} + \frac{7}{256} \sin \frac{3\varphi}{2} - \frac{1}{35} \sin \frac{5\varphi}{2} + \frac{63}{8192} \sin \frac{7\varphi}{2} - \frac{2}{231} \sin \frac{9\varphi}{2} \\
\phi_{0,3,6} &= -\frac{15}{512} \sin \frac{\varphi}{2} + \frac{5}{168} \sin \frac{3\varphi}{2} - \frac{63}{4096} \sin \frac{5\varphi}{2} + \frac{5}{308} \sin \frac{7\varphi}{2} - \frac{231}{65536} \sin \frac{9\varphi}{2} \\
&\quad + \frac{4}{1001} \sin \frac{11\varphi}{2} \\
\phi_{0,3,7} &= -\frac{5}{192} \sin \frac{\varphi}{2} + \frac{81}{4096} \sin \frac{3\varphi}{2} - \frac{25}{1232} \sin \frac{5\varphi}{2} + \frac{693}{81920} \sin \frac{7\varphi}{2} - \frac{9}{1001} \sin \frac{9\varphi}{2} \\
&\quad + \frac{429}{262144} \sin \frac{11\varphi}{2} - \frac{4}{2145} \sin \frac{13\varphi}{2} \\
\phi_{0,3,8} &= -\frac{21}{1024} \sin \frac{\varphi}{2} + \frac{175}{8448} \sin \frac{3\varphi}{2} - \frac{2079}{163840} \sin \frac{5\varphi}{2} + \frac{15}{1144} \sin \frac{7\varphi}{2} - \frac{3003}{655360} \sin \frac{9\varphi}{2} \\
&\quad + \frac{7}{1430} \sin \frac{11\varphi}{2} - \frac{6435}{8388608} \sin \frac{13\varphi}{2} + \frac{32}{36465} \sin \frac{15\varphi}{2} \\
\phi_{0,3,9} &= -\frac{105}{5632} \sin \frac{\varphi}{2} + \frac{77}{5120} \sin \frac{3\varphi}{2} - \frac{35}{2288} \sin \frac{5\varphi}{2} + \frac{1287}{163840} \sin \frac{7\varphi}{2} - \frac{7}{858} \sin \frac{9\varphi}{2} \\
&\quad + \frac{1287}{524288} \sin \frac{11\varphi}{2} - \frac{32}{12155} \sin \frac{13\varphi}{2} + \frac{12155}{33554432} \sin \frac{15\varphi}{2} - \frac{96}{230945} \sin \frac{17\varphi}{2}
\end{aligned}$$

$$\begin{aligned}
\phi_{2,3,0} &= -\frac{1}{10} \sin \frac{3\varphi}{2} \\
\phi_{2,3,1} &= \frac{3}{40} \sin \frac{\varphi}{2} + \frac{11}{140} \sin \frac{5\varphi}{2} \\
\phi_{2,3,2} &= \frac{3}{64} \sin \frac{\varphi}{2} - \frac{7}{80} \sin \frac{3\varphi}{2} - \frac{41}{840} \sin \frac{7\varphi}{2} \\
\phi_{2,3,3} &= \frac{65}{896} \sin \frac{\varphi}{2} - \frac{7}{256} \sin \frac{3\varphi}{2} + \frac{145}{2016} \sin \frac{5\varphi}{2} + \frac{103}{3696} \sin \frac{9\varphi}{2} \\
\phi_{2,3,4} &= \frac{27}{512} \sin \frac{\varphi}{2} - \frac{17959}{241920} \sin \frac{3\varphi}{2} + \frac{63}{4096} \sin \frac{5\varphi}{2} - \frac{2263}{44352} \sin \frac{7\varphi}{2} - \frac{489}{32032} \sin \frac{11\varphi}{2} \\
\phi_{2,3,5} &= \frac{8929}{138240} \sin \frac{\varphi}{2} - \frac{291}{8192} \sin \frac{3\varphi}{2} + \frac{2483}{39424} \sin \frac{5\varphi}{2} - \frac{693}{81920} \sin \frac{7\varphi}{2} + \frac{12829}{384384} \sin \frac{9\varphi}{2} + \frac{1121}{137280} \sin \frac{13\varphi}{2} \\
\phi_{2,3,6} &= \frac{4151}{81920} \sin \frac{\varphi}{2} - \frac{1358089}{21288960} \sin \frac{3\varphi}{2} + \frac{3729}{163840} \sin \frac{5\varphi}{2} - \frac{63161}{1317888} \sin \frac{7\varphi}{2} + \frac{3003}{655360} \sin \frac{9\varphi}{2} \\
&\quad - \frac{34169}{1647360} \sin \frac{11\varphi}{2} - \frac{20081}{4667520} \sin \frac{15\varphi}{2} \\
\phi_{2,3,7} &= \frac{2161361}{37847040} \sin \frac{\varphi}{2} - \frac{21857}{589824} \sin \frac{3\varphi}{2} + \frac{10216499}{184504320} \sin \frac{5\varphi}{2} - \frac{18447}{1310720} \sin \frac{7\varphi}{2} + \frac{1342447}{39536640} \sin \frac{9\varphi}{2} \\
&\quad - \frac{1287}{524288} \sin \frac{11\varphi}{2} + \frac{63367}{5091840} \sin \frac{13\varphi}{2} + \frac{132641}{59121920} \sin \frac{17\varphi}{2}
\end{aligned}$$

$$\begin{aligned}
\phi_{4,3,0} &= \frac{1}{280} \sin \frac{3\varphi}{2} \\
\phi_{4,3,1} &= -\frac{1}{224} \sin \frac{\varphi}{2} - \frac{13 \sin \frac{5\varphi}{2}}{2520} \\
\phi_{4,3,2} &= -\frac{1}{256} \sin \frac{\varphi}{2} + \frac{7}{864} \sin \frac{3\varphi}{2} + \frac{109 \sin \frac{7\varphi}{2}}{22176} \\
\phi_{4,3,3} &= -\frac{59 \sin \frac{\varphi}{2}}{6912} + \frac{3 \sin \frac{3\varphi}{2}}{1024} - \frac{25 \sin \frac{5\varphi}{2}}{2688} - \frac{17 \sin \frac{9\varphi}{2}}{4368} \\
\phi_{4,3,4} &= -\frac{23 \sin \frac{\varphi}{2}}{3072} + \frac{17431 \sin \frac{3\varphi}{2}}{1520640} - \frac{33 \sin \frac{5\varphi}{2}}{16384} + \frac{355 \sin \frac{7\varphi}{2}}{41184} + \frac{457 \sin \frac{11\varphi}{2}}{164736} \\
\phi_{4,3,5} &= -\frac{7781 \sin \frac{\varphi}{2}}{675840} + \frac{869 \sin \frac{3\varphi}{2}}{147456} - \frac{120277 \sin \frac{5\varphi}{2}}{9884160} + \frac{429 \sin \frac{7\varphi}{2}}{327680} - \frac{13919 \sin \frac{9\varphi}{2}}{1976832} \\
&\quad - \frac{25879 \sin \frac{13\varphi}{2}}{14002560}
\end{aligned}$$

$$\begin{aligned}
\phi_{6,3,0} &= -\frac{\sin \frac{3\varphi}{2}}{15120} \\
\phi_{6,3,1} &= \frac{\sin \frac{\varphi}{2}}{8640} + \frac{\sin \frac{5\varphi}{2}}{7392} \\
\phi_{6,3,2} &= \frac{\sin \frac{\varphi}{2}}{7680} - \frac{53 \sin \frac{3\varphi}{2}}{190080} - \frac{7 \sin \frac{7\varphi}{2}}{41184} \\
\phi_{6,3,3} &= \frac{61 \sin \frac{\varphi}{2}}{168960} - \frac{11 \sin \frac{3\varphi}{2}}{92160} + \frac{997 \sin \frac{5\varphi}{2}}{2471040} + \frac{19 \sin \frac{9\varphi}{2}}{112320}
\end{aligned}$$

$$\begin{aligned}
\phi_{8,3,0} &= \frac{\sin \frac{3\varphi}{2}}{1330560} \\
\phi_{8,3,1} &= -\frac{\sin \frac{\varphi}{2}}{591360} - \frac{17 \sin \frac{5\varphi}{2}}{8648640}
\end{aligned}$$

1.3. Third singular exponent ($j = 5$).

$$\begin{aligned}
\phi_{0,5,0} &= \sin \frac{5\varphi}{2} \\
\phi_{0,5,1} &= -\frac{1}{4} \sin \frac{3\varphi}{2} \\
\phi_{0,5,2} &= \frac{3}{32} \sin \frac{\varphi}{2} + \frac{3}{28} \sin \frac{5\varphi}{2} \\
\phi_{0,5,3} &= \frac{5}{128} \sin \frac{\varphi}{2} - \frac{9}{112} \sin \frac{3\varphi}{2} - \frac{1}{21} \sin \frac{7\varphi}{2} \\
\phi_{0,5,4} &= \frac{45}{896} \sin \frac{\varphi}{2} - \frac{35}{2048} \sin \frac{3\varphi}{2} + \frac{3}{56} \sin \frac{5\varphi}{2} + \frac{5}{231} \sin \frac{9\varphi}{2} \\
\phi_{0,5,5} &= \frac{15}{512} \sin \frac{\varphi}{2} - \frac{5}{112} \sin \frac{3\varphi}{2} + \frac{63}{8192} \sin \frac{5\varphi}{2} - \frac{5}{154} \sin \frac{7\varphi}{2} - \frac{10}{1001} \sin \frac{11\varphi}{2} \\
\phi_{0,5,6} &= \frac{25}{768} \sin \frac{\varphi}{2} - \frac{135}{8192} \sin \frac{3\varphi}{2} + \frac{125}{3696} \sin \frac{5\varphi}{2} - \frac{231}{65536} \sin \frac{7\varphi}{2} + \frac{75}{4004} \sin \frac{9\varphi}{2} + \frac{2}{429} \sin \frac{13\varphi}{2} \\
\phi_{0,5,7} &= \frac{45}{2048} \sin \frac{\varphi}{2} - \frac{125}{4224} \sin \frac{3\varphi}{2} + \frac{297}{32768} \sin \frac{5\varphi}{2} - \frac{375}{16016} \sin \frac{7\varphi}{2} + \frac{429}{262144} \sin \frac{9\varphi}{2} \\
&\quad - \frac{3}{286} \sin \frac{11\varphi}{2} - \frac{16}{7293} \sin \frac{15\varphi}{2} \\
\phi_{0,5,8} &= \frac{525}{22528} \sin \frac{\varphi}{2} - \frac{231}{16384} \sin \frac{3\varphi}{2} + \frac{875}{36608} \sin \frac{5\varphi}{2} - \frac{1287}{262144} \sin \frac{7\varphi}{2} + \frac{35}{2288} \sin \frac{9\varphi}{2} \\
&\quad - \frac{6435}{8388608} \sin \frac{11\varphi}{2} + \frac{14}{2431} \sin \frac{13\varphi}{2} + \frac{48}{46189} \sin \frac{17\varphi}{2} \\
\\
\phi_{2,5,0} &= -\frac{1}{14} \sin \frac{5\varphi}{2} \\
\phi_{2,5,1} &= \frac{3}{56} \sin \frac{3\varphi}{2} + \frac{5}{84} \sin \frac{7\varphi}{2} \\
\phi_{2,5,2} &= -\frac{15}{448} \sin \frac{\varphi}{2} - \frac{67 \sin \frac{5\varphi}{2}}{1008} - \frac{71 \sin \frac{9\varphi}{2}}{1848} \\
\phi_{2,5,3} &= -\frac{5}{256} \sin \frac{\varphi}{2} + \frac{445 \sin \frac{3\varphi}{2}}{8064} + \frac{1261 \sin \frac{7\varphi}{2}}{22176} + \frac{361 \sin \frac{11\varphi}{2}}{16016} \\
\phi_{2,5,4} &= -\frac{185 \sin \frac{\varphi}{2}}{4608} + \frac{45 \sin \frac{3\varphi}{2}}{4096} - \frac{31309 \sin \frac{5\varphi}{2}}{532224} - \frac{725 \sin \frac{9\varphi}{2}}{17472} - \frac{173 \sin \frac{13\varphi}{2}}{13728} \\
\phi_{2,5,5} &= -\frac{665 \sin \frac{\varphi}{2}}{24576} + \frac{15589 \sin \frac{3\varphi}{2}}{304128} - \frac{99 \sin \frac{5\varphi}{2}}{16384} + \frac{33853 \sin \frac{7\varphi}{2}}{658944} + \frac{4567 \sin \frac{11\varphi}{2}}{164736} + \frac{3197 \sin \frac{15\varphi}{2}}{466752} \\
\phi_{2,5,6} &= -\frac{21763 \sin \frac{\varphi}{2}}{540672} + \frac{1705 \sin \frac{3\varphi}{2}}{98304} - \frac{961937 \sin \frac{5\varphi}{2}}{18450432} + \frac{429 \sin \frac{7\varphi}{2}}{131072} - \frac{263249 \sin \frac{9\varphi}{2}}{6589440} \\
&\quad - \frac{5771 \sin \frac{13\varphi}{2}}{329472} - \frac{21613 \sin \frac{17\varphi}{2}}{5912192} \\
\\
\phi_{4,5,0} &= \frac{1}{504} \sin \frac{5\varphi}{2} \\
\phi_{4,5,1} &= -\frac{5 \sin \frac{3\varphi}{2}}{2016} - \frac{17 \sin \frac{7\varphi}{2}}{5544} \\
\phi_{4,5,2} &= \frac{5 \sin \frac{\varphi}{2}}{2304} + \frac{10 \sin \frac{5\varphi}{2}}{2079} + \frac{295 \sin \frac{9\varphi}{2}}{96096} \\
\phi_{4,5,3} &= \frac{5 \sin \frac{\varphi}{2}}{3072} - \frac{35 \sin \frac{3\varphi}{2}}{6912} - \frac{955 \sin \frac{7\varphi}{2}}{164736} - \frac{1}{396} \sin \frac{11\varphi}{2} \\
\phi_{4,5,4} &= \frac{25 \sin \frac{\varphi}{2}}{5632} - \frac{55 \sin \frac{3\varphi}{2}}{49152} + \frac{28211 \sin \frac{5\varphi}{2}}{3953664} + \frac{919 \sin \frac{9\varphi}{2}}{164736} + \frac{5197 \sin \frac{13\varphi}{2}}{2800512}
\end{aligned}$$

$$\begin{aligned}
\phi_{6,5,0} &= -\frac{\sin \frac{5\varphi}{2}}{33264} \\
\phi_{6,5,1} &= \frac{\sin \frac{3\varphi}{2}}{19008} + \frac{19 \sin \frac{7\varphi}{2}}{288288} \\
\phi_{6,5,2} &= -\frac{\sin \frac{\varphi}{2}}{16896} - \frac{67 \sin \frac{5\varphi}{2}}{494208} - \frac{\sin \frac{9\varphi}{2}}{11440} \\
\phi_{8,5,0} &= \frac{\sin \frac{5\varphi}{2}}{3459456}
\end{aligned}$$

1.4. Fourth singular exponent ($j = 7$).

$$\begin{aligned}
\phi_{0,7,0} &= \sin \frac{7\varphi}{2} \\
\phi_{0,7,1} &= -\frac{1}{4} \sin \frac{5\varphi}{2} \\
\phi_{0,7,2} &= \frac{3}{32} \sin \frac{3\varphi}{2} + \frac{1}{9} \sin \frac{7\varphi}{2} \\
\phi_{0,7,3} &= -\frac{5}{128} \sin \frac{\varphi}{2} - \frac{1}{12} \sin \frac{5\varphi}{2} - \frac{5}{99} \sin \frac{9\varphi}{2} \\
\phi_{0,7,4} &= -\frac{35 \sin \frac{\varphi}{2}}{2048} + \frac{5}{96} \sin \frac{3\varphi}{2} + \frac{5}{88} \sin \frac{7\varphi}{2} + \frac{10}{429} \sin \frac{11\varphi}{2} \\
\phi_{0,7,5} &= -\frac{35 \sin \frac{\varphi}{2}}{1152} + \frac{63 \sin \frac{3\varphi}{2}}{8192} - \frac{25}{528} \sin \frac{5\varphi}{2} - \frac{5}{143} \sin \frac{9\varphi}{2} - \frac{14 \sin \frac{13\varphi}{2}}{1287} \\
\phi_{0,7,6} &= -\frac{35 \sin \frac{\varphi}{2}}{2048} + \frac{875 \sin \frac{3\varphi}{2}}{25344} - \frac{231 \sin \frac{5\varphi}{2}}{65536} + \frac{125 \sin \frac{7\varphi}{2}}{3432} + \frac{35 \sin \frac{11\varphi}{2}}{1716} + \frac{112 \sin \frac{15\varphi}{2}}{21879} \\
\phi_{0,7,7} &= -\frac{525 \sin \frac{\varphi}{2}}{22528} + \frac{77 \sin \frac{3\varphi}{2}}{8192} - \frac{875 \sin \frac{5\varphi}{2}}{27456} + \frac{429 \sin \frac{7\varphi}{2}}{262144} - \frac{175 \sin \frac{9\varphi}{2}}{6864} - \frac{28 \sin \frac{13\varphi}{2}}{2431} - \frac{112 \sin \frac{17\varphi}{2}}{46189} \\
\\
\phi_{2,7,0} &= -\frac{1}{18} \sin \frac{7\varphi}{2} \\
\phi_{2,7,1} &= \frac{1}{24} \sin \frac{5\varphi}{2} + \frac{19}{396} \sin \frac{9\varphi}{2} \\
\phi_{2,7,2} &= -\frac{5}{192} \sin \frac{3\varphi}{2} - \frac{85 \sin \frac{7\varphi}{2}}{1584} - \frac{109 \sin \frac{11\varphi}{2}}{3432} \\
\phi_{2,7,3} &= \frac{35 \sin \frac{\varphi}{2}}{2304} + \frac{565 \sin \frac{5\varphi}{2}}{12672} + \frac{647 \sin \frac{9\varphi}{2}}{13728} + \frac{391 \sin \frac{13\varphi}{2}}{20592} \\
\phi_{2,7,4} &= \frac{35 \sin \frac{\varphi}{2}}{4096} - \frac{1645 \sin \frac{3\varphi}{2}}{50688} - \frac{48259 \sin \frac{7\varphi}{2}}{988416} - \frac{321 \sin \frac{11\varphi}{2}}{9152} - \frac{7543 \sin \frac{15\varphi}{2}}{700128} \\
\phi_{2,7,5} &= \frac{5915 \sin \frac{\varphi}{2}}{270336} - \frac{77 \sin \frac{3\varphi}{2}}{16384} + \frac{168343 \sin \frac{5\varphi}{2}}{3953664} + \frac{430007 \sin \frac{9\varphi}{2}}{9884160} + \frac{66637 \sin \frac{13\varphi}{2}}{2800512} + \frac{17517 \sin \frac{17\varphi}{2}}{2956096} \\
\\
\phi_{4,7,0} &= \frac{1}{792} \sin \frac{7\varphi}{2} \\
\phi_{4,7,1} &= -\frac{5 \sin \frac{5\varphi}{2}}{3168} - \frac{7 \sin \frac{9\varphi}{2}}{3432} \\
\phi_{4,7,2} &= \frac{35 \sin \frac{3\varphi}{2}}{25344} + \frac{395 \sin \frac{7\varphi}{2}}{123552} + \frac{29 \sin \frac{11\varphi}{2}}{13728} \\
\phi_{4,7,3} &= -\frac{35 \sin \frac{\varphi}{2}}{33792} - \frac{3325 \sin \frac{5\varphi}{2}}{988416} - \frac{179 \sin \frac{9\varphi}{2}}{44928} - \frac{625 \sin \frac{13\varphi}{2}}{350064}
\end{aligned}$$

$$\begin{aligned}\phi_{6,7,0} &= -\frac{\sin \frac{7\varphi}{2}}{61776} \\ \phi_{6,7,1} &= \frac{7 \sin \frac{5\varphi}{2}}{247104} + \frac{23 \sin \frac{9\varphi}{2}}{617760}\end{aligned}$$

1.5. Fifth singular exponent ($j = 9$).

$$\begin{aligned}\phi_{0,9,0} &= \sin \frac{9\varphi}{2} \\ \phi_{0,9,1} &= -\frac{1}{4} \sin \frac{7\varphi}{2} \\ \phi_{0,9,2} &= \frac{3}{32} \sin \frac{5\varphi}{2} + \frac{5}{44} \sin \frac{9\varphi}{2} \\ \phi_{0,9,3} &= -\frac{5}{128} \sin \frac{3\varphi}{2} - \frac{15}{176} \sin \frac{7\varphi}{2} - \frac{15}{286} \sin \frac{11\varphi}{2} \\ \phi_{0,9,4} &= \frac{35 \sin \frac{\varphi}{2}}{2048} + \frac{75 \sin \frac{5\varphi}{2}}{1408} + \frac{135 \sin \frac{9\varphi}{2}}{2288} + \frac{7}{286} \sin \frac{13\varphi}{2} \\ \phi_{0,9,5} &= \frac{63 \sin \frac{\varphi}{2}}{8192} - \frac{175 \sin \frac{3\varphi}{2}}{5632} - \frac{225 \sin \frac{7\varphi}{2}}{4576} - \frac{21}{572} \sin \frac{11\varphi}{2} - \frac{28 \sin \frac{15\varphi}{2}}{2431} \\ \phi_{0,9,6} &= \frac{1575 \sin \frac{\varphi}{2}}{90112} - \frac{231 \sin \frac{3\varphi}{2}}{65536} + \frac{2625 \sin \frac{5\varphi}{2}}{73216} + \frac{175 \sin \frac{9\varphi}{2}}{4576} + \frac{105 \sin \frac{13\varphi}{2}}{4862} + \frac{252 \sin \frac{17\varphi}{2}}{46189}\end{aligned}$$

$$\begin{aligned}\phi_{2,9,0} &= -\frac{1}{22} \sin \frac{9\varphi}{2} \\ \phi_{2,9,1} &= \frac{3}{88} \sin \frac{7\varphi}{2} + \frac{23}{572} \sin \frac{11\varphi}{2} \\ \phi_{2,9,2} &= -\frac{15}{704} \sin \frac{5\varphi}{2} - \frac{103 \sin \frac{9\varphi}{2}}{2288} - \frac{31 \sin \frac{13\varphi}{2}}{1144} \\ \phi_{2,9,3} &= \frac{35 \sin \frac{3\varphi}{2}}{2816} + \frac{685 \sin \frac{7\varphi}{2}}{18304} + \frac{553 \sin \frac{11\varphi}{2}}{13728} + \frac{639 \sin \frac{15\varphi}{2}}{38896} \\ \phi_{2,9,4} &= -\frac{315 \sin \frac{\varphi}{2}}{45056} - \frac{1995 \sin \frac{5\varphi}{2}}{73216} - \frac{5293 \sin \frac{9\varphi}{2}}{126720} - \frac{14197 \sin \frac{13\varphi}{2}}{466752} - \frac{13933 \sin \frac{17\varphi}{2}}{1478048}\end{aligned}$$

$$\begin{aligned}\phi_{4,9,0} &= \frac{\sin \frac{9\varphi}{2}}{1144} \\ \phi_{4,9,1} &= -\frac{5 \sin \frac{7\varphi}{2}}{4576} - \frac{5 \sin \frac{11\varphi}{2}}{3432} \\ \phi_{4,9,2} &= \frac{35 \sin \frac{5\varphi}{2}}{36608} + \frac{47 \sin \frac{9\varphi}{2}}{20592} + \frac{361 \sin \frac{13\varphi}{2}}{233376}\end{aligned}$$

$$\phi_{6,9,0} = -\frac{\sin \frac{9\varphi}{2}}{102960}$$

1.6. Sixth singular exponent ($j = 11$).

$$\begin{aligned}
\phi_{0,11,0} &= \sin \frac{11\varphi}{2} \\
\phi_{0,11,1} &= -\frac{1}{4} \sin \frac{9\varphi}{2} \\
\phi_{0,11,2} &= \frac{3}{32} \sin \frac{7\varphi}{2} + \frac{3}{26} \sin \frac{11\varphi}{2} \\
\phi_{0,11,3} &= -\frac{5}{128} \sin \frac{5\varphi}{2} - \frac{9}{104} \sin \frac{9\varphi}{2} - \frac{7}{130} \sin \frac{13\varphi}{2} \\
\phi_{0,11,4} &= \frac{35 \sin \frac{3\varphi}{2}}{2048} + \frac{45}{832} \sin \frac{7\varphi}{2} + \frac{63 \sin \frac{11\varphi}{2}}{1040} + \frac{28 \sin \frac{15\varphi}{2}}{1105} \\
\phi_{0,11,5} &= -\frac{63 \sin \frac{\varphi}{2}}{8192} - \frac{105 \sin \frac{5\varphi}{2}}{3328} - \frac{21}{416} \sin \frac{9\varphi}{2} - \frac{42 \sin \frac{13\varphi}{2}}{1105} - \frac{252 \sin \frac{17\varphi}{2}}{20995} \\
\\
\phi_{2,11,0} &= -\frac{1}{26} \sin \frac{11\varphi}{2} \\
\phi_{2,11,1} &= \frac{3}{104} \sin \frac{9\varphi}{2} + \frac{9}{260} \sin \frac{13\varphi}{2} \\
\phi_{2,11,2} &= -\frac{15}{832} \sin \frac{7\varphi}{2} - \frac{121 \sin \frac{11\varphi}{2}}{3120} - \frac{209 \sin \frac{15\varphi}{2}}{8840} \\
\phi_{2,11,3} &= \frac{35 \sin \frac{5\varphi}{2}}{3328} + \frac{161 \sin \frac{9\varphi}{2}}{4992} + \frac{3733 \sin \frac{13\varphi}{2}}{106080} + \frac{4867 \sin \frac{17\varphi}{2}}{335920} \\
\\
\phi_{4,11,0} &= \frac{\sin \frac{11\varphi}{2}}{1560} \\
\phi_{4,11,1} &= -\frac{\sin \frac{9\varphi}{2}}{1248} - \frac{29 \sin \frac{13\varphi}{2}}{26520}
\end{aligned}$$

1.7. Seventh singular exponent ($j = 13$).

$$\begin{aligned}
\phi_{0,13,0} &= \sin \frac{13\varphi}{2} \\
\phi_{0,13,1} &= -\frac{1}{4} \sin \frac{11\varphi}{2} \\
\phi_{0,13,2} &= \frac{3}{32} \sin \frac{9\varphi}{2} + \frac{7}{60} \sin \frac{13\varphi}{2} \\
\phi_{0,13,3} &= -\frac{5}{128} \sin \frac{7\varphi}{2} - \frac{7}{80} \sin \frac{11\varphi}{2} - \frac{14}{255} \sin \frac{15\varphi}{2} \\
\phi_{0,13,4} &= \frac{35 \sin \frac{5\varphi}{2}}{2048} + \frac{7}{128} \sin \frac{9\varphi}{2} + \frac{21}{340} \sin \frac{13\varphi}{2} + \frac{42 \sin \frac{17\varphi}{2}}{1615} \\
\\
\phi_{2,13,0} &= -\frac{1}{30} \sin \frac{13\varphi}{2} \\
\phi_{2,13,1} &= \frac{1}{40} \sin \frac{11\varphi}{2} + \frac{31 \sin \frac{15\varphi}{2}}{1020} \\
\phi_{2,13,2} &= -\frac{1}{64} \sin \frac{9\varphi}{2} - \frac{139 \sin \frac{13\varphi}{2}}{4080} - \frac{271 \sin \frac{17\varphi}{2}}{12920} \\
\\
\phi_{4,13,0} &= \frac{\sin \frac{13\varphi}{2}}{2040}
\end{aligned}$$

1.8. **Higher order singular exponent** ($j = 15, 17, 19, 21$).

$$\begin{aligned}
 \phi_{0,15,0} &= \sin \frac{15\varphi}{2} \\
 \phi_{0,15,1} &= -\frac{1}{4} \sin \frac{13\varphi}{2} \\
 \phi_{0,15,2} &= \frac{3}{32} \sin \frac{11\varphi}{2} + \frac{2}{17} \sin \frac{15\varphi}{2} \\
 \phi_{0,15,3} &= -\frac{5}{128} \sin \frac{9\varphi}{2} - \frac{3}{34} \sin \frac{13\varphi}{2} - \frac{18}{323} \sin \frac{17\varphi}{2}
 \end{aligned}$$

$$\begin{aligned}
 \phi_{2,15,0} &= -\frac{1}{34} \sin \frac{15\varphi}{2} \\
 \phi_{2,15,1} &= \frac{3}{136} \sin \frac{13\varphi}{2} + \frac{35 \sin \frac{17\varphi}{2}}{1292}
 \end{aligned}$$

$$\begin{aligned}
 \phi_{0,17,0} &= \sin \frac{17\varphi}{2} \\
 \phi_{0,17,1} &= -\frac{1}{4} \sin \frac{15\varphi}{2} \\
 \phi_{0,17,2} &= \frac{3}{32} \sin \frac{13\varphi}{2} + \frac{9}{76} \sin \frac{17\varphi}{2}
 \end{aligned}$$

$$\phi_{2,17,0} = \frac{1}{38} \sin \frac{17\varphi}{2}$$

$$\begin{aligned}
 \phi_{0,19,0} &= \sin \frac{19\varphi}{2} \\
 \phi_{0,19,1} &= -\frac{1}{4} \sin \frac{17\varphi}{2}
 \end{aligned}$$

$$\phi_{0,21,0} = \sin \frac{21\varphi}{2}$$

2. DUAL SINGULAR FUNCTIONS AND DUAL-SHADOWS FOR THE CRACK $-\pi \leq \varphi \leq \pi$ 2.1. First singular exponent ($j = 1$).

$$\begin{aligned}
\psi_{0,1,0} &= \sin \frac{\varphi}{2} \\
\psi_{0,1,1} &= -\frac{1}{4} \sin \frac{3\varphi}{2} \\
\psi_{0,1,2} &= \frac{3}{32} \sin \frac{5\varphi}{2} \\
\psi_{0,1,3} &= -\frac{5}{128} \sin \frac{7\varphi}{2} \\
\psi_{0,1,4} &= \frac{35}{2048} \sin \frac{9\varphi}{2}
\end{aligned}$$

$$\begin{aligned}
\psi_{2,1,0} &= -\frac{1}{2} \sin \frac{\varphi}{2} \\
\psi_{2,1,1} &= -\frac{1}{4} \sin \frac{\varphi}{2} + \frac{3}{8} \sin \frac{3\varphi}{2} \\
\psi_{2,1,2} &= -\frac{13}{48} \sin \frac{\varphi}{2} + \frac{1}{8} \sin \frac{3\varphi}{2} - \frac{15}{64} \sin \frac{5\varphi}{2} \\
\psi_{2,1,3} &= -\frac{17}{96} \sin \frac{\varphi}{2} + \frac{85}{384} \sin \frac{3\varphi}{2} - \frac{1}{16} \sin \frac{5\varphi}{2} + \frac{35}{256} \sin \frac{7\varphi}{2} \\
\psi_{2,1,4} &= -\frac{2059 \sin \frac{\varphi}{2}}{11520} + \frac{7}{64} \sin \frac{3\varphi}{2} - \frac{245 \sin \frac{5\varphi}{2}}{1536} + \frac{1}{32} \sin \frac{7\varphi}{2} - \frac{315 \sin \frac{9\varphi}{2}}{4096}
\end{aligned}$$

$$\begin{aligned}
\psi_{4,1,0} &= \frac{1}{24} \sin \frac{\varphi}{2} \\
\psi_{4,1,1} &= \frac{1}{24} \sin \frac{\varphi}{2} - \frac{5}{96} \sin \frac{3\varphi}{2} \\
\psi_{4,1,2} &= \frac{19}{288} \sin \frac{\varphi}{2} - \frac{1}{32} \sin \frac{3\varphi}{2} + \frac{35}{768} \sin \frac{5\varphi}{2} \\
\psi_{4,1,3} &= \frac{23}{384} \sin \frac{\varphi}{2} - \frac{161 \sin \frac{3\varphi}{2}}{2304} + \frac{1}{48} \sin \frac{5\varphi}{2} - \frac{35 \sin \frac{7\varphi}{2}}{1024} \\
\psi_{4,1,4} &= \frac{3431 \sin \frac{\varphi}{2}}{46080} - \frac{3}{64} \sin \frac{3\varphi}{2} + \frac{63 \sin \frac{5\varphi}{2}}{1024} - \frac{5}{384} \sin \frac{7\varphi}{2} + \frac{385 \sin \frac{9\varphi}{2}}{16384}
\end{aligned}$$

2.2. Second singular exponent ($j = 3$).

$$\begin{aligned}
\psi_{0,3,0} &= \sin\left(\frac{3\varphi}{2}\right) \\
\psi_{0,3,1} &= -\frac{1}{4}\sin\left(\frac{5\varphi}{2}\right) \\
\psi_{0,3,2} &= \frac{1}{4}\sin\left(\frac{3\varphi}{2}\right) + \frac{3}{32}\sin\left(\frac{7\varphi}{2}\right) \\
\psi_{0,3,3} &= -\frac{3}{16}\sin\left(\frac{5\varphi}{2}\right) - \frac{5}{128}\sin\left(\frac{9\varphi}{2}\right) \\
\psi_{0,3,4} &= \frac{15}{128}\sin\left(\frac{7\varphi}{2}\right) + \frac{35}{2048}\sin\left(\frac{11\varphi}{2}\right)
\end{aligned}$$

$$\begin{aligned}
\psi_{2,3,0} &= \frac{1}{2}\sin\left(\frac{3\varphi}{2}\right) \\
\psi_{2,3,1} &= \frac{1}{4}\sin\left(\frac{\varphi}{2}\right) - \frac{3}{8}\sin\left(\frac{5\varphi}{2}\right) \\
\psi_{2,3,2} &= \frac{1}{8}\sin\left(\frac{\varphi}{2}\right) - \frac{5}{16}\sin\left(\frac{3\varphi}{2}\right) + \frac{15}{64}\sin\left(\frac{7\varphi}{2}\right) \\
\psi_{2,3,3} &= \frac{19}{96}\sin\left(\frac{\varphi}{2}\right) - \frac{1}{16}\sin\left(\frac{3\varphi}{2}\right) + \frac{35}{128}\sin\left(\frac{5\varphi}{2}\right) - \frac{35}{256}\sin\left(\frac{9\varphi}{2}\right) \\
\psi_{2,3,4} &= \frac{23}{192}\sin\left(\frac{\varphi}{2}\right) - \frac{491\sin\left(\frac{3\varphi}{2}\right)}{2304} + \frac{1}{32}\sin\left(\frac{5\varphi}{2}\right) - \frac{105}{512}\sin\left(\frac{7\varphi}{2}\right) + \frac{315\sin\left(\frac{11\varphi}{2}\right)}{4096}
\end{aligned}$$

$$\begin{aligned}
\psi_{4,3,0} &= -\frac{1}{8}\sin\left(\frac{3\varphi}{2}\right) \\
\psi_{4,3,1} &= \frac{1}{24}\sin\left(\frac{\varphi}{2}\right) + \frac{5}{32}\sin\left(\frac{5\varphi}{2}\right) \\
\psi_{4,3,2} &= \frac{1}{96}\sin\left(\frac{\varphi}{2}\right) - \frac{5}{72}\sin\left(\frac{3\varphi}{2}\right) - \frac{35}{256}\sin\left(\frac{7\varphi}{2}\right) \\
\psi_{4,3,3} &= \frac{3}{128}\sin\left(\frac{\varphi}{2}\right) + \frac{175\sin\left(\frac{5\varphi}{2}\right)}{2304} + \frac{105\sin\left(\frac{9\varphi}{2}\right)}{1024} \\
\psi_{4,3,4} &= \frac{1}{384}\sin\left(\frac{\varphi}{2}\right) - \frac{1477\sin\left(\frac{3\varphi}{2}\right)}{46080} - \frac{1}{384}\sin\left(\frac{5\varphi}{2}\right) - \frac{35}{512}\sin\left(\frac{7\varphi}{2}\right) - \frac{1155\sin\left(\frac{11\varphi}{2}\right)}{16384}
\end{aligned}$$

2.3. Third singular exponent ($j = 5$).

$$\begin{aligned}
\psi_{0,5,0} &= \sin\left(\frac{5\varphi}{2}\right) \\
\psi_{0,5,1} &= -\frac{1}{4}\sin\left(\frac{7\varphi}{2}\right) \\
\psi_{0,5,2} &= \frac{1}{6}\sin\left(\frac{5\varphi}{2}\right) + \frac{3}{32}\sin\left(\frac{9\varphi}{2}\right) \\
\psi_{0,5,3} &= -\frac{1}{6}\sin\left(\frac{3\varphi}{2}\right) - \frac{1}{8}\sin\left(\frac{7\varphi}{2}\right) - \frac{5}{128}\sin\left(\frac{11\varphi}{2}\right) \\
\psi_{0,5,4} &= \frac{3}{16}\sin\left(\frac{5\varphi}{2}\right) + \frac{5}{64}\sin\left(\frac{9\varphi}{2}\right) + \frac{35}{2048}\sin\left(\frac{13\varphi}{2}\right)
\end{aligned}$$

$$\begin{aligned}
\psi_{2,5,0} &= \frac{1}{6}\sin\left(\frac{5\varphi}{2}\right) \\
\psi_{2,5,1} &= -\frac{5}{12}\sin\left(\frac{3\varphi}{2}\right) - \frac{1}{8}\sin\left(\frac{7\varphi}{2}\right) \\
\psi_{2,5,2} &= -\frac{1}{8}\sin\left(\frac{\varphi}{2}\right) + \frac{23}{48}\sin\left(\frac{5\varphi}{2}\right) + \frac{5}{64}\sin\left(\frac{9\varphi}{2}\right) \\
\psi_{2,5,3} &= -\frac{1}{16}\sin\left(\frac{\varphi}{2}\right) + \frac{7}{32}\sin\left(\frac{3\varphi}{2}\right) - \frac{155}{384}\sin\left(\frac{7\varphi}{2}\right) - \frac{35}{768}\sin\left(\frac{11\varphi}{2}\right) \\
\psi_{2,5,4} &= -\frac{25}{192}\sin\left(\frac{\varphi}{2}\right) + \frac{1}{32}\sin\left(\frac{3\varphi}{2}\right) - \frac{559\sin\left(\frac{5\varphi}{2}\right)}{2304} + \frac{455\sin\left(\frac{9\varphi}{2}\right)}{1536} + \frac{105\sin\left(\frac{13\varphi}{2}\right)}{4096}
\end{aligned}$$

$$\begin{aligned}
\psi_{4,5,0} &= \frac{1}{24}\sin\left(\frac{5\varphi}{2}\right) \\
\psi_{4,5,1} &= \frac{1}{8}\sin\left(\frac{3\varphi}{2}\right) - \frac{5}{96}\sin\left(\frac{7\varphi}{2}\right) \\
\psi_{4,5,2} &= -\frac{5}{96}\sin\left(\frac{\varphi}{2}\right) - \frac{55}{288}\sin\left(\frac{5\varphi}{2}\right) + \frac{35}{768}\sin\left(\frac{9\varphi}{2}\right) \\
\psi_{4,5,3} &= -\frac{1}{48}\sin\left(\frac{\varphi}{2}\right) + \frac{115\sin\left(\frac{3\varphi}{2}\right)}{1152} + \frac{455\sin\left(\frac{7\varphi}{2}\right)}{2304} - \frac{35\sin\left(\frac{11\varphi}{2}\right)}{1024} \\
\psi_{4,5,4} &= -\frac{7}{144}\sin\left(\frac{\varphi}{2}\right) + \frac{1}{128}\sin\left(\frac{3\varphi}{2}\right) - \frac{1141\sin\left(\frac{5\varphi}{2}\right)}{9216} - \frac{175\sin\left(\frac{9\varphi}{2}\right)}{1024} + \frac{385\sin\left(\frac{13\varphi}{2}\right)}{16384}
\end{aligned}$$

3. PRIMAL FUNCTIONS AND SHADOWS FOR 90° V-NOTCH $-\pi \leq \varphi \leq \pi/2$ 3.1. First singular exponent ($j = 1$).

$$\begin{aligned}
\phi_{0,1,0} &= \sin \frac{2\varphi}{3} + \frac{1}{\sqrt{3}} \cos \frac{2\varphi}{3} \\
\phi_{0,1,1} &= \frac{1}{4} \sin \frac{\varphi}{3} - \frac{1}{4\sqrt{3}} \cos \frac{\varphi}{3} + \frac{1}{20} \sin \frac{5\varphi}{3} + \frac{1}{20\sqrt{3}} \cos \frac{5\varphi}{3} \\
\phi_{0,1,2} &= \frac{\sqrt{3}}{40} \cos \frac{2\varphi}{3} + \frac{3}{40} \sin \frac{2\varphi}{3} + \frac{\sqrt{3}}{32} \cos \frac{4\varphi}{3} - \frac{3}{32} \sin \frac{4\varphi}{3} \\
\phi_{0,1,3} &= -\frac{13\sqrt{3}}{640} \cos \frac{\varphi}{3} + \frac{39}{640} \sin \frac{\varphi}{3} - \frac{13\sqrt{3}}{1280} \cos \frac{5\varphi}{3} - \frac{39}{1280} \sin \frac{5\varphi}{3} - \frac{5}{128\sqrt{3}} \cos \frac{7\varphi}{3} \\
&\quad + \frac{5}{128} \sin \frac{7\varphi}{3} + \frac{7}{1280} \sin \frac{11\varphi}{3} + \frac{7}{1280\sqrt{3}} \cos \frac{11\varphi}{3} \\
\phi_{0,1,4} &= \frac{247\sqrt{3}}{20480} \cos \frac{2\varphi}{3} + \frac{741}{20480} \sin \frac{2\varphi}{3} + \frac{5}{128\sqrt{3}} \cos \frac{4\varphi}{3} - \frac{5}{128} \sin \frac{4\varphi}{3} + \frac{83}{7040\sqrt{3}} \cos \frac{8\varphi}{3} \\
&\quad + \frac{83}{7040} \sin \frac{8\varphi}{3} + \frac{35}{2048\sqrt{3}} \cos \frac{10\varphi}{3} - \frac{35}{2048} \sin \frac{10\varphi}{3} \\
\phi_{0,1,5} &= -\frac{169\sqrt{3}}{16384} \cos \frac{\varphi}{3} + \frac{507}{16384} \sin \frac{\varphi}{3} - \frac{6021\sqrt{3}}{901120} \cos \frac{5\varphi}{3} - \frac{18063}{901120} \sin \frac{5\varphi}{3} - \frac{63\sqrt{3}}{8192} \cos \frac{7\varphi}{3} \\
&\quad + \frac{189}{8192} \sin \frac{7\varphi}{3} - \frac{415}{78848\sqrt{3}} \cos \frac{11\varphi}{3} - \frac{415}{78848} \sin \frac{11\varphi}{3} - \frac{21\sqrt{3}}{8192} \cos \frac{13\varphi}{3} + \frac{63}{8192} \sin \frac{13\varphi}{3} \\
&\quad + \frac{109}{157696\sqrt{3}} \cos \frac{17\varphi}{3} + \frac{109}{157696} \sin \frac{17\varphi}{3} \\
\phi_{0,1,6} &= \frac{2609\sqrt{3}}{360448} \cos \frac{2\varphi}{3} + \frac{1001\sqrt{3}}{131072} \cos \frac{4\varphi}{3} + \frac{36745\sqrt{3}}{10092544} \cos \frac{8\varphi}{3} + \frac{357\sqrt{3}}{81920} \cos \frac{10\varphi}{3} \\
&\quad + \frac{1137\sqrt{3}}{1531904} \cos \frac{14\varphi}{3} + \frac{77\sqrt{3}}{65536} \cos \frac{16\varphi}{3} + \frac{7827}{360448} \sin \frac{2\varphi}{3} - \frac{3003}{131072} \sin \frac{4\varphi}{3} \\
&\quad + \frac{110235}{10092544} \sin \frac{8\varphi}{3} - \frac{1071}{81920} \sin \frac{10\varphi}{3} + \frac{3411}{1531904} \sin \frac{14\varphi}{3} - \frac{231}{65536} \sin \frac{16\varphi}{3} \\
\phi_{0,1,7} &= -\frac{18641\sqrt{3}}{2883584} \cos \frac{\varphi}{3} - \frac{575075}{40370176\sqrt{3}} \cos \frac{5\varphi}{3} - \frac{1365\sqrt{3}}{262144} \cos \frac{7\varphi}{3} - \frac{1330111\sqrt{3}}{686292992} \cos \frac{11\varphi}{3} \\
&\quad - \frac{3157\sqrt{3}}{1310720} \cos \frac{13\varphi}{3} - \frac{42069\sqrt{3}}{122552320} \cos \frac{17\varphi}{3} - \frac{143\sqrt{3}}{262144} \cos \frac{19\varphi}{3} + \frac{55923}{2883584} \sin \frac{\varphi}{3} \\
&\quad - \frac{575075}{40370176} \sin \frac{5\varphi}{3} + \frac{4095}{262144} \sin \frac{7\varphi}{3} - \frac{3990333}{686292992} \sin \frac{11\varphi}{3} + \frac{9471}{1310720} \sin \frac{13\varphi}{3} \\
&\quad - \frac{126207}{122552320} \sin \frac{17\varphi}{3} + \frac{429}{262144} \sin \frac{19\varphi}{3} + \frac{87217}{857866240} \sin \frac{23\varphi}{3} + \frac{87217}{857866240\sqrt{3}} \cos \frac{23\varphi}{3} \\
\phi_{0,1,8} &= \frac{1373975}{92274688\sqrt{3}} \cos \frac{2\varphi}{3} + \frac{74361\sqrt{3}}{14417920} \cos \frac{4\varphi}{3} + \frac{437105}{49020928\sqrt{3}} \cos \frac{8\varphi}{3} + \frac{141141\sqrt{3}}{41943040} \cos \frac{10\varphi}{3} \\
&\quad + \frac{56281699\sqrt{3}}{54903439360} \cos \frac{14\varphi}{3} + \frac{429\sqrt{3}}{327680} \cos \frac{16\varphi}{3} + \frac{8995579}{19730923520\sqrt{3}} \cos \frac{20\varphi}{3} + \frac{2145\sqrt{3}}{8388608} \cos \frac{22\varphi}{3} \\
&\quad + \frac{1373975}{92274688} \sin \frac{2\varphi}{3} - \frac{223083}{14417920} \sin \frac{4\varphi}{3} + \frac{437105}{49020928} \sin \frac{8\varphi}{3} - \frac{423423}{41943040} \sin \frac{10\varphi}{3} \\
&\quad + \frac{168845097}{54903439360} \sin \frac{14\varphi}{3} - \frac{1287}{327680} \sin \frac{16\varphi}{3} + \frac{8995579}{19730923520} \sin \frac{20\varphi}{3} - \frac{6435}{8388608} \sin \frac{22\varphi}{3} \\
\phi_{0,1,9} &= -\frac{1674225\sqrt{3}}{369098752} \cos \frac{\varphi}{3} - \frac{22357035\sqrt{3}}{6274678784} \cos \frac{5\varphi}{3} - \frac{640381\sqrt{3}}{167772160} \cos \frac{7\varphi}{3} - \frac{79139337\sqrt{3}}{43922751488} \cos \frac{11\varphi}{3} \\
&\quad - \frac{351351\sqrt{3}}{167772160} \cos \frac{13\varphi}{3} - \frac{2140689\sqrt{3}}{3992977408} \cos \frac{17\varphi}{3} - \frac{23595\sqrt{3}}{33554432} \cos \frac{19\varphi}{3} \\
&\quad - \frac{62969053}{293145149440\sqrt{3}} \cos \frac{23\varphi}{3} - \frac{12155}{33554432\sqrt{3}} \cos \frac{25\varphi}{3} + \frac{5022675}{369098752} \sin \frac{\varphi}{3} - \frac{67071105}{6274678784} \sin \frac{5\varphi}{3} \\
&\quad + \frac{1921143}{167772160} \sin \frac{7\varphi}{3} - \frac{237418011}{43922751488} \sin \frac{11\varphi}{3} + \frac{1054053}{167772160} \sin \frac{13\varphi}{3} - \frac{6422067}{3992977408} \sin \frac{17\varphi}{3} \\
&\quad + \frac{70785}{33554432} \sin \frac{19\varphi}{3} - \frac{62969053}{293145149440} \sin \frac{23\varphi}{3} + \frac{12155}{33554432} \sin \frac{25\varphi}{3} + \frac{187981}{11152261120} \sin \frac{29\varphi}{3} + \frac{187981}{11152261120\sqrt{3}} \cos \frac{29\varphi}{3} \\
\phi_{0,1,10} &= \frac{2710107\sqrt{3}}{738197504} \cos \frac{2\varphi}{3} + \frac{507745\sqrt{3}}{134217728} \cos \frac{4\varphi}{3} + \frac{61145349\sqrt{3}}{25098715136} \cos \frac{8\varphi}{3} + \frac{898079\sqrt{3}}{335544320} \cos \frac{10\varphi}{3} \\
&\quad + \frac{2152824343\sqrt{3}}{2020446568448} \cos \frac{14\varphi}{3} + \frac{340197\sqrt{3}}{268435456} \cos \frac{16\varphi}{3} + \frac{20970810791\sqrt{3}}{75045158256640} \cos \frac{20\varphi}{3} + \frac{75361}{67108864\sqrt{3}} \cos \frac{22\varphi}{3} \\
&\quad + \frac{1787693753}{18310297026560\sqrt{3}} \cos \frac{26\varphi}{3} + \frac{46189}{268435456\sqrt{3}} \cos \frac{28\varphi}{3} + \frac{8130321}{738197504} \sin \frac{2\varphi}{3} - \frac{1523235}{134217728} \sin \frac{4\varphi}{3} + \frac{183436047}{25098715136} \sin \frac{8\varphi}{3} \\
&\quad - \frac{2694237}{335544320} \sin \frac{10\varphi}{3} + \frac{6458473029}{2020446568448} \sin \frac{14\varphi}{3} - \frac{1020591}{268435456} \sin \frac{16\varphi}{3} + \frac{62912432373}{75045158256640} \sin \frac{20\varphi}{3} - \frac{75361}{67108864} \sin \frac{22\varphi}{3} \\
&\quad + \frac{1787693753}{18310297026560} \sin \frac{26\varphi}{3} - \frac{46189}{268435456} \sin \frac{28\varphi}{3}
\end{aligned}$$

$$\begin{aligned}
\phi_{2,1,0} &= \frac{1}{20} \left(-\sqrt{3} \cos \frac{2\varphi}{3} - 3 \sin \frac{2\varphi}{3} \right) \\
\phi_{2,1,1} &= \frac{3}{80} \sqrt{3} \cos \frac{\varphi}{3} + \frac{11}{320} \sqrt{3} \cos \frac{5\varphi}{3} + \frac{67\sqrt{3} \cos \frac{11\varphi}{3}}{3520} - \frac{9}{80} \sin \frac{\varphi}{3} + \frac{33}{320} \sin \frac{5\varphi}{3} + \frac{201 \sin \frac{11\varphi}{3}}{3520} \\
\phi_{2,1,2} &= -\frac{99\sqrt{3} \cos \frac{2\varphi}{3}}{2560} - \frac{3}{128} \sqrt{3} \cos \frac{4\varphi}{3} - \frac{87\sqrt{3} \cos \frac{8\varphi}{3}}{3520} - \frac{297 \sin \frac{2\varphi}{3}}{2560} + \frac{9}{128} \sin \frac{4\varphi}{3} - \frac{261 \sin \frac{8\varphi}{3}}{3520} \\
\phi_{2,1,3} &= \frac{33\sqrt{3} \cos \frac{\varphi}{3}}{1024} + \frac{177\sqrt{3} \cos \frac{5\varphi}{3}}{5632} + \frac{7}{512} \sqrt{3} \cos \frac{7\varphi}{3} + \frac{10207\sqrt{3} \cos \frac{11\varphi}{3}}{788480} + \frac{38427\sqrt{3} \cos \frac{17\varphi}{3}}{13404160} \\
&\quad - \frac{99 \sin \frac{\varphi}{3}}{1024} + \frac{531 \sin \frac{5\varphi}{3}}{5632} - \frac{21}{512} \sin \frac{7\varphi}{3} + \frac{30621 \sin \frac{11\varphi}{3}}{788480} + \frac{115281 \sin \frac{17\varphi}{3}}{13404160} \\
\phi_{2,1,4} &= -\frac{28387\sqrt{3} \cos \frac{2\varphi}{3}}{901120} - \frac{385\sqrt{3} \cos \frac{4\varphi}{3}}{16384} - \frac{136329\sqrt{3} \cos \frac{8\varphi}{3}}{6307840} - \frac{63\sqrt{3} \cos \frac{10\varphi}{3}}{8192} - \frac{709\sqrt{3} \cos \frac{14\varphi}{3}}{95744} \\
&\quad - \frac{85161 \sin \frac{2\varphi}{3}}{901120} + \frac{1155 \sin \frac{4\varphi}{3}}{16384} - \frac{408987 \sin \frac{8\varphi}{3}}{6307840} + \frac{189 \sin \frac{10\varphi}{3}}{8192} - \frac{2127 \sin \frac{14\varphi}{3}}{95744} \\
\phi_{2,1,5} &= \frac{195993\sqrt{3} \cos \frac{\varphi}{3}}{7208960} + \frac{163131\sqrt{3} \cos \frac{5\varphi}{3}}{6307840} + \frac{2079\sqrt{3} \cos \frac{7\varphi}{3}}{131072} + \frac{1200027\sqrt{3} \cos \frac{11\varphi}{3}}{85786624} + \frac{693\sqrt{3} \cos \frac{13\varphi}{3}}{163840} + \frac{4022021\sqrt{3} \cos \frac{17\varphi}{3}}{1072332800} \\
&\quad + \frac{11088127\sqrt{3} \cos \frac{23\varphi}{3}}{24663654400} - \frac{587979 \sin \frac{\varphi}{3}}{7208960} + \frac{489393 \sin \frac{5\varphi}{3}}{6307840} - \frac{6237 \sin \frac{7\varphi}{3}}{131072} + \frac{3600081 \sin \frac{11\varphi}{3}}{85786624} - \frac{2079 \sin \frac{13\varphi}{3}}{163840} + \frac{12066063 \sin \frac{17\varphi}{3}}{1072332800} \\
&\quad + \frac{33264381 \sin \frac{23\varphi}{3}}{24663654400} \\
\phi_{2,1,6} &= -\frac{2611551\sqrt{3} \cos \frac{2\varphi}{3}}{100925440} - \frac{613557\sqrt{3} \cos \frac{4\varphi}{3}}{28835840} - \frac{9448749\sqrt{3} \cos \frac{8\varphi}{3}}{490209280} - \frac{53361\sqrt{3} \cos \frac{10\varphi}{3}}{5242880} - \frac{293357739\sqrt{3} \cos \frac{14\varphi}{3}}{34314649600} - \frac{3003\sqrt{3} \cos \frac{16\varphi}{3}}{1310720} \\
&\quad - \frac{1164771\sqrt{3} \cos \frac{20\varphi}{3}}{580321280} - \frac{7834653 \sin \frac{2\varphi}{3}}{100925440} + \frac{1840671 \sin \frac{4\varphi}{3}}{28835840} - \frac{28346247 \sin \frac{8\varphi}{3}}{490209280} + \frac{160083 \sin \frac{10\varphi}{3}}{5242880} - \frac{880073217 \sin \frac{14\varphi}{3}}{34314649600} + \frac{9009 \sin \frac{16\varphi}{3}}{1310720} \\
&\quad - \frac{3494313 \sin \frac{20\varphi}{3}}{580321280} \\
\phi_{2,1,7} &= \frac{37211697\sqrt{3} \cos \frac{\varphi}{3}}{1614807040} + \frac{120244237\sqrt{3} \cos \frac{5\varphi}{3}}{5490343936} + \frac{81543\sqrt{3} \cos \frac{7\varphi}{3}}{5242880} + \frac{1843285679\sqrt{3} \cos \frac{11\varphi}{3}}{137258598400} + \frac{33033\sqrt{3} \cos \frac{13\varphi}{3}}{5242880} \\
&\quad + \frac{16043532597\sqrt{3} \cos \frac{17\varphi}{3}}{3156947763200} + \frac{1287\sqrt{3} \cos \frac{19\varphi}{3}}{1048576} + \frac{2067082063\sqrt{3} \cos \frac{23\varphi}{3}}{2052016046080} + \frac{22910746871\sqrt{3} \cos \frac{29\varphi}{3}}{297542326681600} - \frac{111635091 \sin \frac{\varphi}{3}}{1614807040} \\
&\quad + \frac{360732711 \sin \frac{5\varphi}{3}}{5490343936} - \frac{244629 \sin \frac{7\varphi}{3}}{5242880} + \frac{5529857037 \sin \frac{11\varphi}{3}}{137258598400} - \frac{99099 \sin \frac{13\varphi}{3}}{5242880} + \frac{48130597791 \sin \frac{17\varphi}{3}}{3156947763200} \\
&\quad - \frac{3861 \sin \frac{19\varphi}{3}}{1048576} + \frac{6201246189 \sin \frac{23\varphi}{3}}{2052016046080} + \frac{68732240613 \sin \frac{29\varphi}{3}}{297542326681600} \\
\phi_{2,1,8} &= -\frac{23999498037\sqrt{3} \cos \frac{2\varphi}{3}}{1098068787200} - \frac{22191111\sqrt{3} \cos \frac{4\varphi}{3}}{1174405120} - \frac{18777125183\sqrt{3} \cos \frac{8\varphi}{3}}{1098068787200} - \frac{1813851\sqrt{3} \cos \frac{10\varphi}{3}}{167772160} - \frac{225549913639\sqrt{3} \cos \frac{14\varphi}{3}}{25255582105600} \\
&\quad - \frac{127413\sqrt{3} \cos \frac{16\varphi}{3}}{33554432} - \frac{964379391513\sqrt{3} \cos \frac{20\varphi}{3}}{328322567372800} - \frac{21879\sqrt{3} \cos \frac{22\varphi}{3}}{33554432} - \frac{260981373\sqrt{3} \cos \frac{26\varphi}{3}}{497562419200} - \frac{71998494111 \sin \frac{2\varphi}{3}}{1098068787200} \\
&\quad + \frac{66573333 \sin \frac{4\varphi}{3}}{1174405120} - \frac{56331375549 \sin \frac{8\varphi}{3}}{1098068787200} + \frac{5441553 \sin \frac{10\varphi}{3}}{167772160} - \frac{676649740917 \sin \frac{14\varphi}{3}}{25255582105600} + \frac{382239 \sin \frac{16\varphi}{3}}{33554432} \\
&\quad - \frac{2893138174539 \sin \frac{20\varphi}{3}}{328322567372800} + \frac{65637 \sin \frac{22\varphi}{3}}{33554432} - \frac{782944119 \sin \frac{26\varphi}{3}}{497562419200}
\end{aligned}$$

$$\begin{aligned}
\phi_{4,1,0} &= \frac{3\sqrt{3}\cos\frac{2\varphi}{3}}{1280} + \frac{9\sin\frac{2\varphi}{3}}{1280} \\
\phi_{4,1,1} &= -\frac{3\sqrt{3}\cos\frac{\varphi}{3}}{1024} - \frac{171\sqrt{3}\cos\frac{5\varphi}{3}}{56320} - \frac{201\sqrt{3}\cos\frac{11\varphi}{3}}{197120} + \frac{1359\sqrt{3}\cos\frac{17\varphi}{3}}{3351040} + \frac{9\sin\frac{\varphi}{3}}{1024} - \frac{513\sin\frac{5\varphi}{3}}{56320} - \frac{603\sin\frac{11\varphi}{3}}{197120} + \frac{4077\sin\frac{17\varphi}{3}}{3351040} \\
\phi_{4,1,2} &= \frac{27\sqrt{3}\cos\frac{2\varphi}{3}}{5632} + \frac{21\sqrt{3}\cos\frac{4\varphi}{3}}{8192} + \frac{10881\sqrt{3}\cos\frac{8\varphi}{3}}{3153920} + \frac{75\sqrt{3}\cos\frac{14\varphi}{3}}{95744} + \frac{81\sin\frac{2\varphi}{3}}{5632} - \frac{63\sin\frac{4\varphi}{3}}{8192} + \frac{32643\sin\frac{8\varphi}{3}}{3153920} + \frac{225\sin\frac{14\varphi}{3}}{95744} \\
\phi_{4,1,3} &= -\frac{1827\sqrt{3}\cos\frac{\varphi}{3}}{360448} - \frac{14127\sqrt{3}\cos\frac{5\varphi}{3}}{2523136} - \frac{63\sqrt{3}\cos\frac{7\varphi}{3}}{32768} - \frac{127053\sqrt{3}\cos\frac{11\varphi}{3}}{42893312} - \frac{695133\sqrt{3}\cos\frac{17\varphi}{3}}{1072332800} + \frac{2098839\sqrt{3}\cos\frac{23\varphi}{3}}{24663654400} + \frac{5481\sin\frac{\varphi}{3}}{360448} \\
&\quad - \frac{42381\sin\frac{5\varphi}{3}}{2523136} + \frac{189\sin\frac{7\varphi}{3}}{32768} - \frac{381159\sin\frac{11\varphi}{3}}{42893312} - \frac{2085399\sin\frac{17\varphi}{3}}{1072332800} + \frac{6296517\sin\frac{23\varphi}{3}}{24663654400} \\
\phi_{4,1,4} &= \frac{47757\sqrt{3}\cos\frac{2\varphi}{3}}{7208960} + \frac{3213\sqrt{3}\cos\frac{4\varphi}{3}}{720896} + \frac{5907\sqrt{3}\cos\frac{8\varphi}{3}}{1114112} + \frac{693\sqrt{3}\cos\frac{10\varphi}{3}}{524288} + \frac{39272811\sqrt{3}\cos\frac{14\varphi}{3}}{17157324800} + \frac{100143\sqrt{3}\cos\frac{20\varphi}{3}}{246636544} + \frac{143271\sin\frac{2\varphi}{3}}{7208960} \\
&\quad - \frac{9639\sin\frac{4\varphi}{3}}{720896} + \frac{17721\sin\frac{8\varphi}{3}}{1114112} - \frac{2079\sin\frac{10\varphi}{3}}{524288} + \frac{117818433\sin\frac{14\varphi}{3}}{17157324800} + \frac{300429\sin\frac{20\varphi}{3}}{246636544} \\
\phi_{4,1,5} &= -\frac{34371\sqrt{3}\cos\frac{\varphi}{3}}{5242880} - \frac{6843897\sqrt{3}\cos\frac{5\varphi}{3}}{980418560} - \frac{7371\sqrt{3}\cos\frac{7\varphi}{3}}{2097152} - \frac{305405949\sqrt{3}\cos\frac{11\varphi}{3}}{68629299200} - \frac{9009\sqrt{3}\cos\frac{13\varphi}{3}}{10485760} - \frac{362650311\sqrt{3}\cos\frac{17\varphi}{3}}{225496268800} \\
&\quad - \frac{7307871\sqrt{3}\cos\frac{23\varphi}{3}}{28187033600} + \frac{4039929\sqrt{3}\cos\frac{29\varphi}{3}}{260089446400} + \frac{103113\sin\frac{\varphi}{3}}{5242880} - \frac{20531691\sin\frac{5\varphi}{3}}{980418560} + \frac{22113\sin\frac{7\varphi}{3}}{2097152} - \frac{916217847\sin\frac{11\varphi}{3}}{68629299200} + \frac{27027\sin\frac{13\varphi}{3}}{10485760} \\
&\quad - \frac{1087950933\sin\frac{17\varphi}{3}}{225496268800} - \frac{21923613\sin\frac{23\varphi}{3}}{28187033600} + \frac{12119787\sin\frac{29\varphi}{3}}{260089446400} \\
\phi_{4,1,6} &= \frac{10523907\sqrt{3}\cos\frac{2\varphi}{3}}{1372585984} + \frac{484713\sqrt{3}\cos\frac{4\varphi}{3}}{83886080} + \frac{101102037\sqrt{3}\cos\frac{8\varphi}{3}}{15686696960} + \frac{27027\sqrt{3}\cos\frac{10\varphi}{3}}{10485760} + \frac{491571873\sqrt{3}\cos\frac{14\varphi}{3}}{143497625600} + \frac{9009\sqrt{3}\cos\frac{16\varphi}{3}}{16777216} \\
&\quad + \frac{15993341307\sqrt{3}\cos\frac{20\varphi}{3}}{14923753062400} + \frac{3421914141\sqrt{3}\cos\frac{26\varphi}{3}}{22887871283200} + \frac{31571721\sin\frac{2\varphi}{3}}{1372585984} - \frac{1454139\sin\frac{4\varphi}{3}}{83886080} + \frac{303306111\sin\frac{8\varphi}{3}}{15686696960} - \frac{81081\sin\frac{10\varphi}{3}}{10485760} \\
&\quad + \frac{1474715619\sin\frac{14\varphi}{3}}{143497625600} - \frac{27027\sin\frac{16\varphi}{3}}{16777216} + \frac{47980023921\sin\frac{20\varphi}{3}}{14923753062400} + \frac{10265742423\sin\frac{26\varphi}{3}}{22887871283200} \\
\phi_{6,1,0} &= -\frac{3\sqrt{3}\cos\frac{2\varphi}{3}}{56320} - \frac{9\sin\frac{2\varphi}{3}}{56320} \\
\phi_{6,1,1} &= \frac{21\sqrt{3}\cos\frac{\varphi}{3}}{225280} + \frac{39\sqrt{3}\cos\frac{5\varphi}{3}}{394240} + \frac{603\sqrt{3}\cos\frac{11\varphi}{3}}{26808320} \\
&\quad - \frac{4077\sqrt{3}\cos\frac{17\varphi}{3}}{268083200} + \frac{21951\sqrt{3}\cos\frac{23\varphi}{3}}{6165913600} - \frac{63\sin\frac{\varphi}{3}}{225280} + \frac{117\sin\frac{5\varphi}{3}}{394240} + \frac{1809\sin\frac{11\varphi}{3}}{26808320} - \frac{12231\sin\frac{17\varphi}{3}}{268083200} + \frac{65853\sin\frac{23\varphi}{3}}{6165913600} \\
\phi_{6,1,2} &= -\frac{369\sqrt{3}\cos\frac{2\varphi}{3}}{1802240} - \frac{189\sqrt{3}\cos\frac{4\varphi}{3}}{1802240} - \frac{4401\sqrt{3}\cos\frac{8\varphi}{3}}{30638080} \\
&\quad - \frac{963\sqrt{3}\cos\frac{14\varphi}{3}}{38297600} + \frac{7911\sqrt{3}\cos\frac{20\varphi}{3}}{616591360} - \frac{1107\sin\frac{2\varphi}{3}}{1802240} + \frac{567\sin\frac{4\varphi}{3}}{1802240} - \frac{13203\sin\frac{8\varphi}{3}}{30638080} - \frac{2889\sin\frac{14\varphi}{3}}{38297600} + \frac{23733\sin\frac{20\varphi}{3}}{616591360} \\
\phi_{6,1,3} &= \frac{1917\sqrt{3}\cos\frac{\varphi}{3}}{7208960} + \frac{7371\sqrt{3}\cos\frac{5\varphi}{3}}{24510464} + \frac{63\sqrt{3}\cos\frac{7\varphi}{3}}{655360} + \frac{194433\sqrt{3}\cos\frac{11\varphi}{3}}{1225523200} + \frac{4882743\sqrt{3}\cos\frac{17\varphi}{3}}{197309235200} - \frac{4186251\sqrt{3}\cos\frac{23\varphi}{3}}{366431436800} \\
&\quad - \frac{17914173\sqrt{3}\cos\frac{29\varphi}{3}}{14877116334080} - \frac{5751\sin\frac{\varphi}{3}}{7208960} + \frac{22113\sin\frac{5\varphi}{3}}{24510464} - \frac{189\sin\frac{7\varphi}{3}}{655360} + \frac{583299\sin\frac{11\varphi}{3}}{1225523200} + \frac{14648229\sin\frac{17\varphi}{3}}{197309235200} - \frac{12558753\sin\frac{23\varphi}{3}}{366431436800} \\
&\quad - \frac{53742519\sin\frac{29\varphi}{3}}{14877116334080} \\
\phi_{6,1,4} &= -\frac{1041957\sqrt{3}\cos\frac{2\varphi}{3}}{2451046400} - \frac{1449\sqrt{3}\cos\frac{4\varphi}{3}}{5242880} - \frac{1717143\sqrt{3}\cos\frac{8\varphi}{3}}{4902092800} - \frac{819\sqrt{3}\cos\frac{10\varphi}{3}}{10485760} - \frac{238063431\sqrt{3}\cos\frac{14\varphi}{3}}{1578473881600} - \frac{50690457\sqrt{3}\cos\frac{20\varphi}{3}}{2565020057600} \\
&\quad + \frac{12531\sqrt{3}\cos\frac{26\varphi}{3}}{1541478400} - \frac{3125871\sin\frac{2\varphi}{3}}{2451046400} + \frac{4347\sin\frac{4\varphi}{3}}{5242880} - \frac{5151429\sin\frac{8\varphi}{3}}{4902092800} + \frac{2457\sin\frac{10\varphi}{3}}{10485760} - \frac{714190293\sin\frac{14\varphi}{3}}{1578473881600} - \frac{152071371\sin\frac{20\varphi}{3}}{2565020057600} \\
&\quad + \frac{37593\sin\frac{26\varphi}{3}}{1541478400} \\
\phi_{8,1,0} &= \frac{9\sqrt{3}\cos\frac{2\varphi}{3}}{12615680} + \frac{27\sin\frac{2\varphi}{3}}{12615680} \\
\phi_{8,1,1} &= -\frac{81\sqrt{3}\cos\frac{\varphi}{3}}{50462720} - \frac{1467\sqrt{3}\cos\frac{5\varphi}{3}}{857866240} - \frac{603\sqrt{3}\cos\frac{11\varphi}{3}}{2144665600} + \frac{12231\sqrt{3}\cos\frac{17\varphi}{3}}{49327308800} - \frac{65853\sqrt{3}\cos\frac{23\varphi}{3}}{641255014400} + \frac{44469\sqrt{3}\cos\frac{29\varphi}{3}}{2656627916800} + \frac{243\sin\frac{\varphi}{3}}{50462720} \\
&\quad - \frac{4401\sin\frac{5\varphi}{3}}{857866240} - \frac{1809\sin\frac{11\varphi}{3}}{2144665600} + \frac{36693\sin\frac{17\varphi}{3}}{49327308800} - \frac{197559\sin\frac{23\varphi}{3}}{641255014400} + \frac{133407\sin\frac{29\varphi}{3}}{2656627916800} \\
\phi_{8,1,2} &= \frac{7533\sqrt{3}\cos\frac{2\varphi}{3}}{1715732480} + \frac{81\sqrt{3}\cos\frac{4\varphi}{3}}{36700160} + \frac{20277\sqrt{3}\cos\frac{8\varphi}{3}}{6862929920} + \frac{2403\sqrt{3}\cos\frac{14\varphi}{3}}{7046758400} - \frac{405\sqrt{3}\cos\frac{20\varphi}{3}}{1165918208} + \frac{129519\sqrt{3}\cos\frac{26\varphi}{3}}{1430491955200} \\
&\quad + \frac{22599\sin\frac{2\varphi}{3}}{1715732480} - \frac{243\sin\frac{4\varphi}{3}}{36700160} + \frac{60831\sin\frac{8\varphi}{3}}{6862929920} + \frac{7209\sin\frac{14\varphi}{3}}{7046758400} - \frac{1215\sin\frac{20\varphi}{3}}{1165918208} + \frac{388557\sin\frac{26\varphi}{3}}{1430491955200} \\
\phi_{10,1,0} &= -\frac{27\sqrt{3}\cos\frac{2\varphi}{3}}{4289331200} - \frac{81\sin\frac{2\varphi}{3}}{4289331200}
\end{aligned}$$

3.2. Second singular exponent ($j = 2$).

$$\begin{aligned}
\phi_{0,2,0} &= \sin \frac{4\varphi}{3} - \frac{1}{\sqrt{3}} \cos \frac{4\varphi}{3} \\
\phi_{0,2,1} &= \frac{1}{4\sqrt{3}} \cos \frac{\varphi}{3} - \frac{1}{4} \sin \frac{\varphi}{3} - \frac{1}{28} \sin \frac{7\varphi}{3} + \frac{1}{28\sqrt{3}} \cos \frac{7\varphi}{3} \\
\phi_{0,2,2} &= -\frac{\sqrt{3}}{32} \cos \frac{2\varphi}{3} - \frac{\sqrt{3}}{28} \cos \frac{4\varphi}{3} - \frac{3}{32} \sin \frac{2\varphi}{3} + \frac{3}{28} \sin \frac{4\varphi}{3} \\
\phi_{0,2,3} &= \frac{23\sqrt{3}}{896} \cos \frac{\varphi}{3} + \frac{5}{128\sqrt{3}} \cos \frac{5\varphi}{3} + \frac{17\sqrt{3}}{1120} \cos \frac{7\varphi}{3} - \frac{69}{896} \sin \frac{\varphi}{3} + \frac{5}{128} \sin \frac{5\varphi}{3} \\
&\quad - \frac{51}{1120} \sin \frac{7\varphi}{3} - \frac{1}{280} \sin \frac{13\varphi}{3} + \frac{1}{280\sqrt{3}} \cos \frac{13\varphi}{3} \\
\phi_{0,2,4} &= -\frac{85}{1792\sqrt{3}} \cos \frac{2\varphi}{3} - \frac{17\sqrt{3}}{1024} \cos \frac{4\varphi}{3} - \frac{35}{2048\sqrt{3}} \cos \frac{8\varphi}{3} - \frac{35}{1664\sqrt{3}} \cos \frac{10\varphi}{3} \\
&\quad - \frac{85}{1792} \sin \frac{2\varphi}{3} + \frac{51}{1024} \sin \frac{4\varphi}{3} + \frac{35}{2048} \sin \frac{8\varphi}{3} + \frac{35}{1664} \sin \frac{10\varphi}{3} \\
\phi_{0,2,5} &= \frac{391\sqrt{3}}{28672} \cos \frac{\varphi}{3} + \frac{75\sqrt{3}}{8192} \cos \frac{5\varphi}{3} + \frac{531\sqrt{3}}{53248} \cos \frac{7\varphi}{3} + \frac{21\sqrt{3}}{8192} \cos \frac{11\varphi}{3} + \frac{1015}{106496\sqrt{3}} \cos \frac{13\varphi}{3} \\
&\quad - \frac{1173}{28672} \sin \frac{\varphi}{3} + \frac{225}{8192} \sin \frac{5\varphi}{3} - \frac{1593}{53248} \sin \frac{7\varphi}{3} + \frac{63}{8192} \sin \frac{11\varphi}{3} - \frac{1015}{106496} \sin \frac{13\varphi}{3} \\
&\quad - \frac{319}{745472} \sin \frac{19\varphi}{3} + \frac{319}{745472\sqrt{3}} \cos \frac{19\varphi}{3} \\
\phi_{0,2,6} &= -\frac{323\sqrt{3}}{32768} \cos \frac{2\varphi}{3} - \frac{3797\sqrt{3}}{372736} \cos \frac{4\varphi}{3} - \frac{21\sqrt{3}}{4096} \cos \frac{8\varphi}{3} - \frac{19459\sqrt{3}}{3407872} \cos \frac{10\varphi}{3} - \frac{77\sqrt{3}}{65536} \cos \frac{14\varphi}{3} \\
&\quad - \frac{5307\sqrt{3}}{3540992} \cos \frac{16\varphi}{3} - \frac{969}{32768} \sin \frac{2\varphi}{3} + \frac{11391}{372736} \sin \frac{4\varphi}{3} - \frac{63}{4096} \sin \frac{8\varphi}{3} + \frac{58377}{3407872} \sin \frac{10\varphi}{3} \\
&\quad - \frac{231}{65536} \sin \frac{14\varphi}{3} + \frac{15921}{3540992} \sin \frac{16\varphi}{3} \\
\phi_{0,2,7} &= \frac{15023\sqrt{3}}{1703936} \cos \frac{\varphi}{3} + \frac{867\sqrt{3}}{131072} \cos \frac{5\varphi}{3} + \frac{2002421}{95420416\sqrt{3}} \cos \frac{7\varphi}{3} \\
&\quad + \frac{737\sqrt{3}}{262144} \cos \frac{11\varphi}{3} + \frac{5786399\sqrt{3}}{1812987904} \cos \frac{13\varphi}{3} + \frac{143\sqrt{3}}{262144} \cos \frac{17\varphi}{3} + \frac{217587\sqrt{3}}{311607296} \cos \frac{19\varphi}{3} \\
&\quad - \frac{45069}{1703936} \sin \frac{\varphi}{3} + \frac{2601}{131072} \sin \frac{5\varphi}{3} - \frac{2002421}{95420416} \sin \frac{7\varphi}{3} + \frac{2211}{262144} \sin \frac{11\varphi}{3} \\
&\quad - \frac{17359197}{1812987904} \sin \frac{13\varphi}{3} + \frac{429}{262144} \sin \frac{17\varphi}{3} - \frac{652761}{311607296} \sin \frac{19\varphi}{3} - \frac{4139}{68485120} \sin \frac{25\varphi}{3} \\
&\quad + \frac{4139}{68485120\sqrt{3}} \cos \frac{25\varphi}{3} \\
\phi_{0,2,8} &= -\frac{7341\sqrt{3}}{1064960} \cos \frac{2\varphi}{3} - \frac{4616423}{218103808\sqrt{3}} \cos \frac{4\varphi}{3} - \frac{44319\sqrt{3}}{10485760} \cos \frac{8\varphi}{3} - \frac{49508365}{3625975808\sqrt{3}} \cos \frac{10\varphi}{3} \\
&\quad - \frac{5577\sqrt{3}}{3670016} \cos \frac{14\varphi}{3} - \frac{279019063\sqrt{3}}{159542935552} \cos \frac{16\varphi}{3} - \frac{2145\sqrt{3}}{8388608} \cos \frac{20\varphi}{3} - \frac{11327483}{11331174400\sqrt{3}} \cos \frac{22\varphi}{3} \\
&\quad - \frac{22023}{1064960} \sin \frac{2\varphi}{3} + \frac{4616423}{218103808} \sin \frac{4\varphi}{3} - \frac{132957}{10485760} \sin \frac{8\varphi}{3} + \frac{49508365}{3625975808} \sin \frac{10\varphi}{3} \\
&\quad - \frac{16731}{3670016} \sin \frac{14\varphi}{3} + \frac{837057189}{159542935552} \sin \frac{16\varphi}{3} - \frac{6435}{8388608} \sin \frac{20\varphi}{3} + \frac{11327483}{11331174400} \sin \frac{22\varphi}{3} \\
\phi_{0,2,9} &= \frac{5494833\sqrt{3}}{872415232} \cos \frac{\varphi}{3} + \frac{42185\sqrt{3}}{8388608} \cos \frac{5\varphi}{3} + \frac{86863845\sqrt{3}}{16575889408} \cos \frac{7\varphi}{3} + \frac{21879\sqrt{3}}{8388608} \cos \frac{11\varphi}{3} \\
&\quad + \frac{1818260415\sqrt{3}}{638171742208} \cos \frac{13\varphi}{3} + \frac{190905\sqrt{3}}{234881024} \cos \frac{17\varphi}{3} + \frac{15107532969\sqrt{3}}{15954293555200} \cos \frac{19\varphi}{3} \\
&\quad + \frac{12155}{33554432\sqrt{3}} \cos \frac{23\varphi}{3} + \frac{600356599}{1269091532800\sqrt{3}} \cos \frac{25\varphi}{3} - \frac{16484499}{872415232} \sin \frac{\varphi}{3} + \frac{126555}{8388608} \sin \frac{5\varphi}{3} \\
&\quad - \frac{260591535}{16575889408} \sin \frac{7\varphi}{3} + \frac{65637}{8388608} \sin \frac{11\varphi}{3} - \frac{5454781245}{638171742208} \sin \frac{13\varphi}{3} + \frac{572715}{234881024} \sin \frac{17\varphi}{3} \\
&\quad - \frac{45322598907}{15954293555200} \sin \frac{19\varphi}{3} + \frac{12155}{33554432} \sin \frac{23\varphi}{3} - \frac{600356599}{1269091532800} \sin \frac{25\varphi}{3} \\
&\quad - \frac{134768503}{13960006860800} \sin \frac{31\varphi}{3} + \frac{134768503}{13960006860800\sqrt{3}} \cos \frac{31\varphi}{3} \\
\phi_{0,2,10} &= -\frac{18025733\sqrt{3}}{3489660928} \cos \frac{2\varphi}{3} - \frac{87168519\sqrt{3}}{16575889408} \cos \frac{4\varphi}{3} - \frac{102421\sqrt{3}}{29360128} \cos \frac{8\varphi}{3} - \frac{1344919515\sqrt{3}}{364669566976} \cos \frac{10\varphi}{3} \\
&\quad - \frac{736593\sqrt{3}}{469762048} \cos \frac{14\varphi}{3} - \frac{172871033\sqrt{3}}{99714334720} \cos \frac{16\varphi}{3} - \frac{303875}{234881024\sqrt{3}} \cos \frac{20\varphi}{3} - \frac{906362008781\sqrt{3}}{1786880878182400} \cos \frac{22\varphi}{3} \\
&\quad - \frac{46189}{268435456\sqrt{3}} \cos \frac{26\varphi}{3} - \frac{7032308149}{30911443763200\sqrt{3}} \cos \frac{28\varphi}{3} - \frac{54077199}{3489660928} \sin \frac{2\varphi}{3} + \frac{261505557}{16575889408} \sin \frac{4\varphi}{3} - \frac{307263}{29360128} \sin \frac{8\varphi}{3} \\
&\quad + \frac{4034758545}{364669566976} \sin \frac{10\varphi}{3} - \frac{2209779}{469762048} \sin \frac{14\varphi}{3} + \frac{518613099}{99714334720} \sin \frac{16\varphi}{3} - \frac{303875}{234881024} \sin \frac{20\varphi}{3} \\
&\quad + \frac{2719086026343}{1786880878182400} \sin \frac{22\varphi}{3} - \frac{46189}{268435456} \sin \frac{26\varphi}{3} + \frac{7032308149}{30911443763200} \sin \frac{28\varphi}{3}
\end{aligned}$$

$$\begin{aligned}
\phi_{2,2,0} &= \frac{1}{28}\sqrt{3}\cos\frac{4\varphi}{3} - \frac{3}{28}\sin\frac{4\varphi}{3} \\
\phi_{2,2,1} &= -\frac{3}{112}\sqrt{3}\cos\frac{\varphi}{3} - \frac{1}{35}\sqrt{3}\cos\frac{7\varphi}{3} - \frac{97\sqrt{3}\cos\frac{13\varphi}{3}}{7280} + \frac{9}{112}\sin\frac{\varphi}{3} + \frac{3}{35}\sin\frac{7\varphi}{3} + \frac{291\sin\frac{13\varphi}{3}}{7280} \\
\phi_{2,2,2} &= \frac{15}{896}\sqrt{3}\cos\frac{2\varphi}{3} + \frac{141\sqrt{3}\cos\frac{4\varphi}{3}}{4480} + \frac{123\sqrt{3}\cos\frac{10\varphi}{3}}{5824} + \frac{45}{896}\sin\frac{2\varphi}{3} - \frac{423\sin\frac{4\varphi}{3}}{4480} - \frac{369\sin\frac{10\varphi}{3}}{5824} \\
\phi_{2,2,3} &= -\frac{93\sqrt{3}\cos\frac{\varphi}{3}}{3584} - \frac{5}{512}\sqrt{3}\cos\frac{5\varphi}{3} - \frac{633\sqrt{3}\cos\frac{7\varphi}{3}}{23296} \\
&\quad - \frac{21839\sqrt{3}\cos\frac{13\varphi}{3}}{1863680} - \frac{68787\sqrt{3}\cos\frac{19\varphi}{3}}{35409920} + \frac{279\sin\frac{\varphi}{3}}{3584} - \frac{15}{512}\sin\frac{5\varphi}{3} + \frac{1899\sin\frac{7\varphi}{3}}{23296} + \frac{65517\sin\frac{13\varphi}{3}}{1863680} + \frac{206361\sin\frac{19\varphi}{3}}{35409920} \\
\phi_{2,2,4} &= \frac{77\sqrt{3}\cos\frac{2\varphi}{3}}{4096} + \frac{50497\sqrt{3}\cos\frac{4\varphi}{3}}{1863680} + \frac{45\sqrt{3}\cos\frac{8\varphi}{3}}{8192} \\
&\quad + \frac{3207\sqrt{3}\cos\frac{10\varphi}{3}}{163840} + \frac{751\sqrt{3}\cos\frac{16\varphi}{3}}{110656} + \frac{231\sin\frac{2\varphi}{3}}{4096} - \frac{151491\sin\frac{4\varphi}{3}}{1863680} + \frac{135\sin\frac{8\varphi}{3}}{8192} - \frac{9621\sin\frac{10\varphi}{3}}{163840} - \frac{2253\sin\frac{16\varphi}{3}}{110656} \\
\phi_{2,2,5} &= -\frac{49509\sqrt{3}\cos\frac{\varphi}{3}}{2129920} - \frac{207\sqrt{3}\cos\frac{5\varphi}{3}}{16384} - \frac{21369\sqrt{3}\cos\frac{7\varphi}{3}}{917504} - \frac{99\sqrt{3}\cos\frac{11\varphi}{3}}{32768} - \frac{1137957\sqrt{3}\cos\frac{13\varphi}{3}}{87162880} - \frac{22170977\sqrt{3}\cos\frac{19\varphi}{3}}{6232145920} \\
&\quad - \frac{9624067\sqrt{3}\cos\frac{25\varphi}{3}}{31160729600} + \frac{148527\sin\frac{\varphi}{3}}{2129920} - \frac{621\sin\frac{5\varphi}{3}}{16384} + \frac{64107\sin\frac{7\varphi}{3}}{917504} - \frac{297\sin\frac{11\varphi}{3}}{32768} + \frac{3413871\sin\frac{13\varphi}{3}}{87162880} + \frac{66512931\sin\frac{19\varphi}{3}}{6232145920} \\
&\quad + \frac{28872201\sin\frac{25\varphi}{3}}{31160729600} \\
\phi_{2,2,6} &= \frac{5931\sqrt{3}\cos\frac{2\varphi}{3}}{327680} + \frac{2762703\sqrt{3}\cos\frac{4\varphi}{3}}{119275520} + \frac{10593\sqrt{3}\cos\frac{8\varphi}{3}}{1310720} + \frac{32529369\sqrt{3}\cos\frac{10\varphi}{3}}{1812987904} + \frac{429\sqrt{3}\cos\frac{14\varphi}{3}}{262144} \\
&\quad + \frac{816384879\sqrt{3}\cos\frac{16\varphi}{3}}{99714334720} + \frac{54482031\sqrt{3}\cos\frac{22\varphi}{3}}{28327936000} + \frac{17793\sin\frac{2\varphi}{3}}{327680} - \frac{8288109\sin\frac{4\varphi}{3}}{119275520} + \frac{31779\sin\frac{8\varphi}{3}}{1310720} - \frac{97588107\sin\frac{10\varphi}{3}}{1812987904} + \frac{1287\sin\frac{14\varphi}{3}}{262144} \\
&\quad - \frac{2449154637\sin\frac{16\varphi}{3}}{99714334720} - \frac{163446093\sin\frac{22\varphi}{3}}{28327936000} \\
\phi_{2,2,7} &= -\frac{78308991\sqrt{3}\cos\frac{\varphi}{3}}{3816816640} - \frac{449031\sqrt{3}\cos\frac{5\varphi}{3}}{34078720} - \frac{1470654989\sqrt{3}\cos\frac{7\varphi}{3}}{72519516160} - \frac{26169\sqrt{3}\cos\frac{11\varphi}{3}}{5242880} - \frac{1279456777\sqrt{3}\cos\frac{13\varphi}{3}}{99714334720} - \frac{6435\sqrt{3}\cos\frac{17\varphi}{3}}{7340032} \\
&\quad - \frac{4498216551\sqrt{3}\cos\frac{19\varphi}{3}}{906493952000} - \frac{34493067223\sqrt{3}\cos\frac{25\varphi}{3}}{34900017152000} - \frac{58410950531\sqrt{3}\cos\frac{31\varphi}{3}}{1081900531712000} + \frac{234926973\sin\frac{\varphi}{3}}{3816816640} - \frac{1347093\sin\frac{5\varphi}{3}}{34078720} \\
&\quad + \frac{4411964967\sin\frac{7\varphi}{3}}{72519516160} - \frac{78507\sin\frac{11\varphi}{3}}{5242880} + \frac{3838370331\sin\frac{13\varphi}{3}}{99714334720} - \frac{19305\sin\frac{17\varphi}{3}}{7340032} + \frac{13494649653\sin\frac{19\varphi}{3}}{906493952000} + \frac{103479201669\sin\frac{25\varphi}{3}}{34900017152000} \\
&\quad + \frac{175232851593\sin\frac{31\varphi}{3}}{1081900531712000} \\
\phi_{2,2,8} &= \frac{102271257\sqrt{3}\cos\frac{2\varphi}{3}}{6106906624} + \frac{641736027\sqrt{3}\cos\frac{4\varphi}{3}}{31876710400} + \frac{2680227\sqrt{3}\cos\frac{8\varphi}{3}}{293601280} + \frac{10376443733\sqrt{3}\cos\frac{10\varphi}{3}}{638171742208} + \frac{176319\sqrt{3}\cos\frac{14\varphi}{3}}{58720256} \\
&\quad + \frac{36582771103\sqrt{3}\cos\frac{16\varphi}{3}}{4198498304000} + \frac{109395\sqrt{3}\cos\frac{20\varphi}{3}}{234881024} + \frac{3259134726429\sqrt{3}\cos\frac{22\varphi}{3}}{1116800548864000} + \frac{10028501487\sqrt{3}\cos\frac{28\varphi}{3}}{19319652352000} + \frac{306813771\sin\frac{2\varphi}{3}}{6106906624} \\
&\quad - \frac{1925208081\sin\frac{4\varphi}{3}}{31876710400} + \frac{8040681\sin\frac{8\varphi}{3}}{293601280} - \frac{31129331199\sin\frac{10\varphi}{3}}{638171742208} + \frac{528957\sin\frac{14\varphi}{3}}{58720256} - \frac{109748313309\sin\frac{16\varphi}{3}}{4198498304000} + \frac{328185\sin\frac{20\varphi}{3}}{234881024} \\
&\quad - \frac{9777404179287\sin\frac{22\varphi}{3}}{1116800548864000} - \frac{30085504461\sin\frac{28\varphi}{3}}{19319652352000}
\end{aligned}$$

$$\begin{aligned}
\phi_{4,2,0} &= -\frac{3\sqrt{3}\cos\frac{4\varphi}{3}}{2240} + \frac{9\sin\frac{4\varphi}{3}}{2240} \\
\phi_{4,2,1} &= \frac{3\sqrt{3}\cos\frac{\varphi}{3}}{1792} + \frac{45\sqrt{3}\cos\frac{7\varphi}{3}}{23296} + \frac{291\sqrt{3}\cos\frac{13\varphi}{3}}{465920} - \frac{1737\sqrt{3}\cos\frac{19\varphi}{3}}{8852480} - \frac{9\sin\frac{\varphi}{3}}{1792} - \frac{135\sin\frac{7\varphi}{3}}{23296} - \frac{873\sin\frac{13\varphi}{3}}{465920} + \frac{5211\sin\frac{19\varphi}{3}}{8852480} \\
\phi_{4,2,2} &= -\frac{3\sqrt{3}\cos\frac{2\varphi}{3}}{2048} - \frac{141\sqrt{3}\cos\frac{4\varphi}{3}}{46592} - \frac{17181\sqrt{3}\cos\frac{10\varphi}{3}}{7454720} - \frac{111\sqrt{3}\cos\frac{16\varphi}{3}}{221312} - \frac{9\sin\frac{2\varphi}{3}}{2048} + \frac{423\sin\frac{4\varphi}{3}}{46592} + \frac{51543\sin\frac{10\varphi}{3}}{7454720} + \frac{333\sin\frac{16\varphi}{3}}{221312} \\
\phi_{4,2,3} &= \frac{339\sqrt{3}\cos\frac{\varphi}{3}}{106496} + \frac{9\sqrt{3}\cos\frac{5\varphi}{3}}{8192} + \frac{22401\sqrt{3}\cos\frac{7\varphi}{3}}{5963776} + \frac{1161297\sqrt{3}\cos\frac{13\varphi}{3}}{566558720} + \frac{1288377\sqrt{3}\cos\frac{19\varphi}{3}}{3116072960} - \frac{731631\sqrt{3}\cos\frac{25\varphi}{3}}{15580364800} - \frac{1017\sin\frac{\varphi}{3}}{106496} \\
&+ \frac{27\sin\frac{5\varphi}{3}}{8192} - \frac{67203\sin\frac{7\varphi}{3}}{5963776} - \frac{3483891\sin\frac{13\varphi}{3}}{566558720} - \frac{3865131\sin\frac{19\varphi}{3}}{3116072960} + \frac{2194893\sin\frac{25\varphi}{3}}{15580364800} \\
\phi_{4,2,4} &= -\frac{297\sqrt{3}\cos\frac{2\varphi}{3}}{106496} - \frac{75477\sqrt{3}\cos\frac{4\varphi}{3}}{17039360} - \frac{99\sqrt{3}\cos\frac{8\varphi}{3}}{131072} - \frac{210081\sqrt{3}\cos\frac{10\varphi}{3}}{56655872} - \frac{11511609\sqrt{3}\cos\frac{16\varphi}{3}}{7122452480} - \frac{48771\sqrt{3}\cos\frac{22\varphi}{3}}{186368000} - \frac{891\sin\frac{2\varphi}{3}}{106496} \\
&+ \frac{226431\sin\frac{4\varphi}{3}}{17039360} - \frac{297\sin\frac{8\varphi}{3}}{131072} + \frac{630243\sin\frac{10\varphi}{3}}{56655872} + \frac{34534827\sin\frac{16\varphi}{3}}{7122452480} + \frac{146313\sin\frac{22\varphi}{3}}{186368000} \\
\phi_{4,2,5} &= \frac{595161\sqrt{3}\cos\frac{\varphi}{3}}{136314880} + \frac{14949\sqrt{3}\cos\frac{5\varphi}{3}}{6815744} + \frac{12637593\sqrt{3}\cos\frac{7\varphi}{3}}{2589982720} + \frac{1287\sqrt{3}\cos\frac{11\varphi}{3}}{2621440} + \frac{638879919\sqrt{3}\cos\frac{13\varphi}{3}}{199428669440} + \frac{88487757\sqrt{3}\cos\frac{19\varphi}{3}}{76703334400} \\
&+ \frac{2921545173\sqrt{3}\cos\frac{25\varphi}{3}}{17450008576000} - \frac{1085706519\sqrt{3}\cos\frac{31\varphi}{3}}{108190053171200} - \frac{1785483\sin\frac{\varphi}{3}}{136314880} + \frac{44847\sin\frac{5\varphi}{3}}{6815744} - \frac{37912779\sin\frac{7\varphi}{3}}{2589982720} + \frac{3861\sin\frac{11\varphi}{3}}{2621440} \\
&- \frac{1916639757\sin\frac{13\varphi}{3}}{199428669440} - \frac{265463271\sin\frac{19\varphi}{3}}{76703334400} - \frac{8764635519\sin\frac{25\varphi}{3}}{17450008576000} + \frac{3257119557\sin\frac{31\varphi}{3}}{108190053171200} \\
\phi_{4,2,6} &= -\frac{8363223\sqrt{3}\cos\frac{2\varphi}{3}}{2181038080} - \frac{48435399\sqrt{3}\cos\frac{4\varphi}{3}}{9064939520} - \frac{1683\sqrt{3}\cos\frac{8\varphi}{3}}{1048576} - \frac{212164605\sqrt{3}\cos\frac{10\varphi}{3}}{45583695872} - \frac{1287\sqrt{3}\cos\frac{14\varphi}{3}}{4194304} - \frac{25127548689\sqrt{3}\cos\frac{16\varphi}{3}}{9971433472000} \\
&- \frac{652815981\sqrt{3}\cos\frac{22\varphi}{3}}{839699660800} - \frac{1069062327\sqrt{3}\cos\frac{28\varphi}{3}}{11039801344000} - \frac{25089669\sin\frac{2\varphi}{3}}{2181038080} + \frac{145306197\sin\frac{4\varphi}{3}}{9064939520} - \frac{5049\sin\frac{8\varphi}{3}}{1048576} + \frac{636493815\sin\frac{10\varphi}{3}}{45583695872} \\
&- \frac{3861\sin\frac{14\varphi}{3}}{4194304} + \frac{75382646067\sin\frac{16\varphi}{3}}{9971433472000} + \frac{1958447943\sin\frac{22\varphi}{3}}{839699660800} + \frac{3207186981\sin\frac{28\varphi}{3}}{11039801344000} \\
\phi_{6,2,0} &= \frac{3\sqrt{3}\cos\frac{4\varphi}{3}}{116480} - \frac{9\sin\frac{4\varphi}{3}}{116480} \\
\phi_{6,2,1} &= -\frac{3\sqrt{3}\cos\frac{\varphi}{3}}{66560} - \frac{3\sqrt{3}\cos\frac{7\varphi}{3}}{57344} - \frac{873\sqrt{3}\cos\frac{13\varphi}{3}}{70819840} + \frac{5211\sqrt{3}\cos\frac{19\varphi}{3}}{779018240} + \frac{9\sin\frac{\varphi}{3}}{66560} + \frac{9\sin\frac{7\varphi}{3}}{57344} + \frac{2619\sin\frac{13\varphi}{3}}{70819840} - \frac{15633\sin\frac{19\varphi}{3}}{779018240} \\
\phi_{6,2,2} &= \frac{27\sqrt{3}\cos\frac{2\varphi}{3}}{532480} + \frac{459\sqrt{3}\cos\frac{4\varphi}{3}}{4259840} + \frac{9153\sqrt{3}\cos\frac{10\varphi}{3}}{113311744} + \frac{1503\sqrt{3}\cos\frac{16\varphi}{3}}{97377280} - \frac{5211\sqrt{3}\cos\frac{22\varphi}{3}}{856064000} + \frac{81\sin\frac{2\varphi}{3}}{532480} - \frac{1377\sin\frac{4\varphi}{3}}{4259840} - \frac{27459\sin\frac{10\varphi}{3}}{113311744} \\
&- \frac{4509\sin\frac{16\varphi}{3}}{97377280} + \frac{15633\sin\frac{22\varphi}{3}}{856064000} \\
\phi_{6,2,3} &= -\frac{297\sqrt{3}\cos\frac{\varphi}{3}}{2129920} - \frac{99\sqrt{3}\cos\frac{5\varphi}{3}}{2129920} - \frac{2871\sqrt{3}\cos\frac{7\varphi}{3}}{17039360} - \frac{1167711\sqrt{3}\cos\frac{13\varphi}{3}}{12464291840} - \frac{1231299\sqrt{3}\cos\frac{19\varphi}{3}}{77901824000} + \frac{23785659\sqrt{3}\cos\frac{25\varphi}{3}}{4362502144000} \\
&+ \frac{57409293\sqrt{3}\cos\frac{31\varphi}{3}}{67618783232000} + \frac{891\sin\frac{\varphi}{3}}{2129920} - \frac{297\sin\frac{5\varphi}{3}}{2129920} + \frac{8613\sin\frac{7\varphi}{3}}{17039360} + \frac{3503133\sin\frac{13\varphi}{3}}{12464291840} + \frac{3693897\sin\frac{19\varphi}{3}}{77901824000} - \frac{71356977\sin\frac{25\varphi}{3}}{4362502144000} \\
&- \frac{172227879\sin\frac{31\varphi}{3}}{67618783232000} \\
\phi_{6,2,4} &= \frac{19701\sqrt{3}\cos\frac{2\varphi}{3}}{136314880} + \frac{767421\sqrt{3}\cos\frac{4\varphi}{3}}{3237478400} + \frac{99\sqrt{3}\cos\frac{8\varphi}{3}}{2621440} + \frac{293115\sqrt{3}\cos\frac{10\varphi}{3}}{1424490496} + \frac{35369211\sqrt{3}\cos\frac{16\varphi}{3}}{383516672000} + \frac{65479527\sqrt{3}\cos\frac{22\varphi}{3}}{4985716736000} \\
&- \frac{306636441\sqrt{3}\cos\frac{28\varphi}{3}}{77278609408000} + \frac{59103\sin\frac{2\varphi}{3}}{136314880} - \frac{2302263\sin\frac{4\varphi}{3}}{3237478400} + \frac{297\sin\frac{8\varphi}{3}}{2621440} - \frac{879345\sin\frac{10\varphi}{3}}{1424490496} - \frac{106107633\sin\frac{16\varphi}{3}}{383516672000} - \frac{196438581\sin\frac{22\varphi}{3}}{4985716736000} \\
&+ \frac{919909323\sin\frac{28\varphi}{3}}{77278609408000} \\
\phi_{8,2,0} &= -\frac{9\sqrt{3}\cos\frac{4\varphi}{3}}{29818880} + \frac{27\sin\frac{4\varphi}{3}}{29818880} \\
\phi_{8,2,1} &= \frac{81\sqrt{3}\cos\frac{\varphi}{3}}{119275520} + \frac{1773\sqrt{3}\cos\frac{7\varphi}{3}}{2266234880} + \frac{873\sqrt{3}\cos\frac{13\varphi}{3}}{6232145920} - \frac{15633\sqrt{3}\cos\frac{19\varphi}{3}}{155803648000} - \frac{243\sin\frac{\varphi}{3}}{119275520} - \frac{5319\sin\frac{7\varphi}{3}}{2266234880} - \frac{2619\sin\frac{13\varphi}{3}}{6232145920} + \frac{46899\sin\frac{19\varphi}{3}}{155803648000} \\
\phi_{8,2,2} &= -\frac{891\sqrt{3}\cos\frac{2\varphi}{3}}{954204160} - \frac{81\sqrt{3}\cos\frac{4\varphi}{3}}{40468480} - \frac{2619\sqrt{3}\cos\frac{10\varphi}{3}}{1812987904} - \frac{4023\sqrt{3}\cos\frac{16\varphi}{3}}{19475456000} + \frac{1516401\sqrt{3}\cos\frac{22\varphi}{3}}{8725004288000} - \frac{2673\sin\frac{2\varphi}{3}}{954204160} + \frac{243\sin\frac{4\varphi}{3}}{40468480} \\
&+ \frac{7857\sin\frac{10\varphi}{3}}{1812987904} + \frac{12069\sin\frac{16\varphi}{3}}{19475456000} - \frac{4549203\sin\frac{22\varphi}{3}}{8725004288000} \\
\phi_{10,2,0} &= \frac{27\sqrt{3}\cos\frac{4\varphi}{3}}{11331174400} - \frac{81\sin\frac{4\varphi}{3}}{11331174400}
\end{aligned}$$

3.3. Higher order exponents ($j = 3, \dots, 11$).

$$\begin{aligned}
\phi_{0,3,0} &= \cos 2\varphi \\
\phi_{0,3,1} &= -\frac{\cos \varphi}{4} - \frac{1}{12} \cos 3\varphi \\
\phi_{0,3,2} &= \frac{3}{32} + \frac{1}{8} \cos 2\varphi \\
\phi_{0,3,3} &= -\frac{\cos \varphi}{8} - \frac{7}{128} \cos 3\varphi - \frac{1}{128} \cos 5\varphi \\
\phi_{0,3,4} &= \frac{5}{96} + \frac{77 \cos 2\varphi}{1024} + \frac{17}{640} \cos 4\varphi \\
\phi_{0,3,5} &= -\frac{315 \cos \varphi}{4096} - \frac{897 \cos 3\varphi}{20480} - \frac{187 \cos 5\varphi}{15360} - \frac{7 \cos 7\varphi}{7680} \\
\phi_{0,3,6} &= \frac{2205}{65536} + \frac{433 \cos 2\varphi}{8192} + \frac{4037 \cos 4\varphi}{163840} + \frac{843 \cos 6\varphi}{143360} \\
\phi_{0,3,7} &= -\frac{875 \cos \varphi}{16384} - \frac{4521 \cos 3\varphi}{131072} - \frac{62597 \cos 5\varphi}{4587520} - \frac{2529 \cos 7\varphi}{917504} - \frac{349 \cos 9\varphi}{2752512} \\
\phi_{0,3,8} &= \frac{1575}{65536} + \frac{248941 \cos 2\varphi}{6291456} + \frac{59729 \cos 4\varphi}{2752512} + \frac{218139 \cos 6\varphi}{29360128} + \frac{1835 \cos 8\varphi}{1376256} \\
\phi_{0,3,9} &= -\frac{1670823 \cos \varphi}{41943040} - \frac{233745 \cos 3\varphi}{8388608} - \frac{1558145 \cos 5\varphi}{117440512} - \frac{1412281 \cos 7\varphi}{352321536} \\
&\quad - \frac{6973 \cos 9\varphi}{11010048} - \frac{2201 \cos 11\varphi}{110100480} \\
\phi_{2,3,0} &= -\frac{1}{12} \cos 2\varphi \\
\phi_{2,3,1} &= \frac{\cos \varphi}{16} + \frac{7}{96} \cos 3\varphi + \frac{1}{32} \cos 5\varphi \\
\phi_{2,3,2} &= -\frac{5}{128} - \frac{61}{768} \cos 2\varphi - \frac{53}{960} \cos 4\varphi \\
\phi_{2,3,3} &= \frac{45 \cos \varphi}{512} + \frac{137 \cos 3\varphi}{1920} + \frac{487 \cos 5\varphi}{15360} + \frac{71 \cos 7\varphi}{15360} \\
\phi_{2,3,4} &= -\frac{385}{8192} - \frac{15437 \cos 2\varphi}{184320} - \frac{19523 \cos 4\varphi}{368640} - \frac{143 \cos 6\varphi}{7680} \\
\phi_{2,3,5} &= \frac{67949 \cos \varphi}{737280} + \frac{309119 \cos 3\varphi}{4423680} + \frac{372203 \cos 5\varphi}{10321920} + \frac{2447 \cos 7\varphi}{245760} + \frac{2365 \cos 9\varphi}{3096576} \\
\phi_{2,3,6} &= -\frac{61733}{1310720} - \frac{648079 \cos 2\varphi}{7864320} - \frac{2200529 \cos 4\varphi}{41287680} - \frac{844089 \cos 6\varphi}{36700160} - \frac{168421 \cos 8\varphi}{30965760} \\
\phi_{2,3,7} &= \frac{2806877 \cos \varphi}{31457280} + \frac{6434681 \cos 3\varphi}{94371840} + \frac{10053427 \cos 5\varphi}{264241152} + \frac{55988167 \cos 7\varphi}{3963617280} + \frac{2806673 \cos 9\varphi}{990904320} + \frac{301339 \cos 11\varphi}{2179989504} \\
\phi_{4,3,0} &= \frac{1}{384} \cos 2\varphi \\
\phi_{4,3,1} &= -\frac{5 \cos \varphi}{1536} - \frac{31 \cos 3\varphi}{7680} - \frac{1}{768} \cos 5\varphi + \frac{3 \cos 7\varphi}{8960} \\
\phi_{4,3,2} &= \frac{35}{12288} + \frac{29 \cos 2\varphi}{4608} + \frac{923 \cos 4\varphi}{184320} + \frac{29 \cos 6\varphi}{26880} \\
\phi_{4,3,3} &= -\frac{161 \cos \varphi}{18432} - \frac{3611 \cos 3\varphi}{442368} - \frac{23633 \cos 5\varphi}{5160960} - \frac{3077 \cos 7\varphi}{3440640} + \frac{9103 \cos 9\varphi}{92897280} \\
\phi_{4,3,4} &= \frac{189}{32768} + \frac{391699 \cos 2\varphi}{35389440} + \frac{17161 \cos 4\varphi}{2064384} + \frac{67513 \cos 6\varphi}{18350080} + \frac{1325 \cos 8\varphi}{2322432} \\
\phi_{4,3,5} &= -\frac{73213 \cos \varphi}{5242880} - \frac{1681349 \cos 3\varphi}{141557760} - \frac{138697 \cos 5\varphi}{18874368} - \frac{5281387 \cos 7\varphi}{1981808640} - \frac{683371 \cos 9\varphi}{1857945600} + \frac{334729 \cos 11\varphi}{13624934400} \\
\phi_{6,3,0} &= -\frac{\cos 2\varphi}{23040} \\
\phi_{6,3,1} &= \frac{7 \cos \varphi}{92160} + \frac{13 \cos 3\varphi}{138240} + \frac{\cos 5\varphi}{43008} - \frac{3 \cos 7\varphi}{286720} + \frac{43 \cos 9\varphi}{23224320} \\
\phi_{6,3,2} &= -\frac{7}{81920} - \frac{427 \cos 2\varphi}{2211840} - \frac{787 \cos 4\varphi}{5160960} - \frac{9 \cos 6\varphi}{286720} + \frac{53 \cos 8\varphi}{5806080} \\
\phi_{6,3,3} &= \frac{161 \cos \varphi}{491520} + \frac{2801 \cos 3\varphi}{8847360} + \frac{3035 \cos 5\varphi}{16515072} + \frac{2767 \cos 7\varphi}{82575360} - \frac{1571 \cos 9\varphi}{176947200} - \frac{7361 \cos 11\varphi}{4541644800} \\
\phi_{8,3,0} &= \frac{\cos 2\varphi}{2211840} \\
\phi_{8,3,1} &= -\frac{\cos \varphi}{983040} - \frac{11 \cos 3\varphi}{8847360} - \frac{\cos 5\varphi}{4128768} + \frac{\cos 7\varphi}{6881280} - \frac{43 \cos 9\varphi}{928972800} + \frac{\cos 11\varphi}{162201600}
\end{aligned}$$

$$\begin{aligned}
\phi_{0,4,0} &= \sin \frac{8}{3}\varphi + \frac{1}{\sqrt{3}} \cos \frac{8}{3}\varphi \\
\phi_{0,4,1} &= -\frac{1}{4\sqrt{3}} \cos \frac{5\varphi}{3} - \frac{1}{4} \sin \frac{5\varphi}{3} - \frac{5}{44} \sin \frac{11\varphi}{3} - \frac{5}{44\sqrt{3}} \cos \frac{11\varphi}{3} \\
\phi_{0,4,2} &= \frac{\sqrt{3}}{32} \cos \frac{2\varphi}{3} + \frac{3}{32} \sin \frac{2\varphi}{3} + \frac{\sqrt{3}}{22} \cos \frac{8\varphi}{3} + \frac{3}{22} \sin \frac{8\varphi}{3} \\
\phi_{0,4,3} &= -\frac{5}{128\sqrt{3}} \cos \frac{\varphi}{3} + \frac{5}{128} \sin \frac{\varphi}{3} - \frac{43\sqrt{3}}{1408} \cos \frac{5\varphi}{3} - \frac{129}{1408} \sin \frac{5\varphi}{3} \\
&\quad - \frac{25\sqrt{3}}{1232} \cos \frac{11\varphi}{3} - \frac{75}{1232} \sin \frac{11\varphi}{3} - \frac{25}{2464} \sin \frac{17\varphi}{3} - \frac{25}{2464\sqrt{3}} \cos \frac{17\varphi}{3} \\
\phi_{0,4,4} &= \frac{155}{2816\sqrt{3}} \cos \frac{2\varphi}{3} + \frac{155}{2816} \sin \frac{2\varphi}{3} + \frac{35}{2048\sqrt{3}} \cos \frac{4\varphi}{3} - \frac{35}{2048} \sin \frac{4\varphi}{3} + \frac{1675\sqrt{3}}{78848} \cos \frac{8\varphi}{3} \\
&\quad + \frac{5025}{78848} \sin \frac{8\varphi}{3} + \frac{725}{23936\sqrt{3}} \cos \frac{14\varphi}{3} + \frac{725}{23936} \sin \frac{14\varphi}{3} \\
\phi_{0,4,5} &= -\frac{945\sqrt{3}}{90112} \cos \frac{\varphi}{3} + \frac{2835}{90112} \sin \frac{\varphi}{3} - \frac{5375\sqrt{3}}{315392} \cos \frac{5\varphi}{3} - \frac{16125}{315392} \sin \frac{5\varphi}{3} - \frac{21\sqrt{3}}{8192} \cos \frac{7\varphi}{3} \\
&\quad + \frac{63}{8192} \sin \frac{7\varphi}{3} - \frac{72225\sqrt{3}}{5361664} \cos \frac{11\varphi}{3} - \frac{216675}{5361664} \sin \frac{11\varphi}{3} - \frac{5365}{382976\sqrt{3}} \cos \frac{17\varphi}{3} \\
&\quad - \frac{5365}{382976} \sin \frac{17\varphi}{3} - \frac{3133}{2680832} \sin \frac{23\varphi}{3} - \frac{3133}{2680832\sqrt{3}} \cos \frac{23\varphi}{3} \\
\phi_{0,4,6} &= \frac{4375\sqrt{3}}{360448} \cos \frac{2\varphi}{3} + \frac{13125}{360448} \sin \frac{2\varphi}{3} + \frac{525\sqrt{3}}{90112} \cos \frac{4\varphi}{3} - \frac{1575}{90112} \sin \frac{4\varphi}{3} + \frac{5125\sqrt{3}}{382976} \cos \frac{8\varphi}{3} \\
&\quad + \frac{15375}{382976} \sin \frac{8\varphi}{3} + \frac{77\sqrt{3}}{65536} \cos \frac{10\varphi}{3} - \frac{231}{65536} \sin \frac{10\varphi}{3} + \frac{684685\sqrt{3}}{85786624} \cos \frac{14\varphi}{3} \\
&\quad + \frac{2054055}{85786624} \sin \frac{14\varphi}{3} + \frac{140577\sqrt{3}}{61659136} \cos \frac{20\varphi}{3} + \frac{421731}{61659136} \sin \frac{20\varphi}{3} \\
\phi_{0,4,7} &= -\frac{11625\sqrt{3} \cos \frac{\varphi}{3}}{1441792} - \frac{25375\sqrt{3} \cos \frac{5\varphi}{3}}{2228224} - \frac{833\sqrt{3} \cos \frac{7\varphi}{3}}{262144} - \frac{9794825 \cos \frac{11\varphi}{3}}{343146496\sqrt{3}} \\
&\quad - \frac{143\sqrt{3} \cos \frac{13\varphi}{3}}{262144} - \frac{36189151\sqrt{3} \cos \frac{17\varphi}{3}}{7892369408} - \frac{984039\sqrt{3} \cos \frac{23\varphi}{3}}{916078592} - \frac{129905 \cos \frac{29\varphi}{3}}{801568768\sqrt{3}} \\
&\quad + \frac{34875 \sin \frac{\varphi}{3}}{1441792} - \frac{76125 \sin \frac{5\varphi}{3}}{2228224} + \frac{2499 \sin \frac{7\varphi}{3}}{262144} - \frac{9794825 \sin \frac{11\varphi}{3}}{343146496} \\
&\quad + \frac{429 \sin \frac{13\varphi}{3}}{262144} - \frac{108567453 \sin \frac{17\varphi}{3}}{7892369408} - \frac{2952117 \sin \frac{23\varphi}{3}}{916078592} - \frac{129905 \sin \frac{29\varphi}{3}}{801568768} \\
\phi_{0,4,8} &= \frac{53925\sqrt{3} \cos \frac{2\varphi}{3}}{6127616} + \frac{10725\sqrt{3} \cos \frac{4\varphi}{3}}{2097152} + \frac{1305175 \cos \frac{8\varphi}{3}}{46137344\sqrt{3}} + \frac{897\sqrt{3} \cos \frac{10\varphi}{3}}{524288} \\
&\quad + \frac{301061615 \cos \frac{14\varphi}{3}}{15784738816\sqrt{3}} + \frac{2145\sqrt{3} \cos \frac{16\varphi}{3}}{8388608} + \frac{300557929\sqrt{3} \cos \frac{20\varphi}{3}}{117258059776} + \frac{89746385 \cos \frac{26\varphi}{3}}{57219678208\sqrt{3}} \\
&\quad + \frac{161775 \sin \frac{2\varphi}{3}}{6127616} - \frac{32175 \sin \frac{4\varphi}{3}}{2097152} + \frac{1305175 \sin \frac{8\varphi}{3}}{46137344} - \frac{2691 \sin \frac{10\varphi}{3}}{524288} + \frac{301061615 \sin \frac{14\varphi}{3}}{15784738816} \\
&\quad - \frac{6435 \sin \frac{16\varphi}{3}}{8388608} + \frac{901673787 \sin \frac{20\varphi}{3}}{117258059776} + \frac{89746385 \sin \frac{26\varphi}{3}}{57219678208} \\
\phi_{2,4,0} &= -\frac{1}{44} \sqrt{3} \cos \frac{8\varphi}{3} - \frac{3}{44} \sin \frac{8\varphi}{3} \\
\phi_{2,4,1} &= \frac{3}{176} \sqrt{3} \cos \frac{5\varphi}{3} + \frac{13}{616} \sqrt{3} \cos \frac{11\varphi}{3} + \frac{9}{176} \sin \frac{5\varphi}{3} + \frac{39}{616} \sin \frac{11\varphi}{3} \\
\phi_{2,4,2} &= -\frac{15\sqrt{3} \cos \frac{2\varphi}{3}}{1408} - \frac{225\sqrt{3} \cos \frac{8\varphi}{3}}{9856} - \frac{37\sqrt{3} \cos \frac{14\varphi}{3}}{2618} - \frac{45 \sin \frac{2\varphi}{3}}{1408} - \frac{675 \sin \frac{8\varphi}{3}}{9856} - \frac{111 \sin \frac{14\varphi}{3}}{2618} \\
\phi_{2,4,3} &= \frac{35\sqrt{3} \cos \frac{\varphi}{3}}{5632} + \frac{105\sqrt{3} \cos \frac{5\varphi}{3}}{5632} + \frac{13599\sqrt{3} \cos \frac{11\varphi}{3}}{670208} + \frac{5751\sqrt{3} \cos \frac{17\varphi}{3}}{670208} - \frac{105 \sin \frac{\varphi}{3}}{5632} + \frac{315 \sin \frac{5\varphi}{3}}{5632} + \frac{40797 \sin \frac{11\varphi}{3}}{670208} + \frac{17253 \sin \frac{17\varphi}{3}}{670208} \\
\phi_{2,4,4} &= -\frac{55\sqrt{3} \cos \frac{2\varphi}{3}}{4096} - \frac{315\sqrt{3} \cos \frac{4\varphi}{3}}{90112} - \frac{110427\sqrt{3} \cos \frac{8\varphi}{3}}{5361664} - \frac{163539\sqrt{3} \cos \frac{14\varphi}{3}}{10723328} - \frac{302101\sqrt{3} \cos \frac{20\varphi}{3}}{61659136} - \frac{165 \sin \frac{2\varphi}{3}}{4096} + \frac{945 \sin \frac{4\varphi}{3}}{90112} \\
&\quad - \frac{331281 \sin \frac{8\varphi}{3}}{5361664} - \frac{490617 \sin \frac{14\varphi}{3}}{10723328} - \frac{906303 \sin \frac{20\varphi}{3}}{61659136} \\
\phi_{2,4,5} &= \frac{405\sqrt{3} \cos \frac{\varphi}{3}}{45056} + \frac{53\sqrt{3} \cos \frac{5\varphi}{3}}{2992} + \frac{63\sqrt{3} \cos \frac{7\varphi}{3}}{32768} + \frac{990783\sqrt{3} \cos \frac{11\varphi}{3}}{53616640} + \frac{10275141\sqrt{3} \cos \frac{17\varphi}{3}}{986546176} + \frac{155323\sqrt{3} \cos \frac{23\varphi}{3}}{57254912} - \frac{1215 \sin \frac{\varphi}{3}}{45056} \\
&\quad + \frac{159 \sin \frac{5\varphi}{3}}{2992} - \frac{189 \sin \frac{7\varphi}{3}}{32768} + \frac{2972349 \sin \frac{11\varphi}{3}}{53616640} + \frac{30825423 \sin \frac{17\varphi}{3}}{986546176} + \frac{465969 \sin \frac{23\varphi}{3}}{57254912} \\
\phi_{2,4,6} &= -\frac{337743\sqrt{3} \cos \frac{2\varphi}{3}}{24510464} - \frac{1503\sqrt{3} \cos \frac{4\varphi}{3}}{262144} - \frac{15848761\sqrt{3} \cos \frac{8\varphi}{3}}{857866240} - \frac{273\sqrt{3} \cos \frac{10\varphi}{3}}{262144} - \frac{579169167\sqrt{3} \cos \frac{14\varphi}{3}}{39461847040} - \frac{687402189\sqrt{3} \cos \frac{20\varphi}{3}}{102600802304} \\
&\quad - \frac{38987053\sqrt{3} \cos \frac{26\varphi}{3}}{26566279168} - \frac{1013229 \sin \frac{2\varphi}{3}}{24510464} + \frac{4509 \sin \frac{4\varphi}{3}}{262144} - \frac{47546283 \sin \frac{8\varphi}{3}}{857866240} + \frac{819 \sin \frac{10\varphi}{3}}{262144} - \frac{1737507501 \sin \frac{14\varphi}{3}}{39461847040} - \frac{2062206567 \sin \frac{20\varphi}{3}}{102600802304} \\
&\quad - \frac{116961159 \sin \frac{26\varphi}{3}}{26566279168}
\end{aligned}$$

$$\begin{aligned}
\phi_{4,4,0} &= \frac{3\sqrt{3}\cos\frac{8\varphi}{3}}{4928} + \frac{9\sin\frac{8\varphi}{3}}{4928} \\
\phi_{4,4,1} &= -\frac{15\sqrt{3}\cos\frac{5\varphi}{3}}{19712} - \frac{333\sqrt{3}\cos\frac{11\varphi}{3}}{335104} + \frac{597\sqrt{3}\cos\frac{23\varphi}{3}}{1926848} - \frac{45\sin\frac{5\varphi}{3}}{19712} - \frac{999\sin\frac{11\varphi}{3}}{335104} + \frac{1791\sin\frac{23\varphi}{3}}{1926848} \\
\phi_{4,4,2} &= \frac{15\sqrt{3}\cos\frac{2\varphi}{3}}{22528} + \frac{1035\sqrt{3}\cos\frac{8\varphi}{3}}{670208} + \frac{5517\sqrt{3}\cos\frac{14\varphi}{3}}{5361664} - \frac{597\sqrt{3}\cos\frac{20\varphi}{3}}{7707392} + \frac{45\sin\frac{2\varphi}{3}}{22528} + \frac{3105\sin\frac{8\varphi}{3}}{670208} + \frac{16551\sin\frac{14\varphi}{3}}{5361664} - \frac{1791\sin\frac{20\varphi}{3}}{7707392} \\
\phi_{4,4,3} &= -\frac{45\sqrt{3}\cos\frac{\varphi}{3}}{90112} - \frac{225\sqrt{3}\cos\frac{5\varphi}{3}}{139264} - \frac{41163\sqrt{3}\cos\frac{11\varphi}{3}}{21446656} - \frac{414027\sqrt{3}\cos\frac{17\varphi}{3}}{493273088} + \frac{4179\sqrt{3}\cos\frac{23\varphi}{3}}{114509824} + \frac{375033\sqrt{3}\cos\frac{29\varphi}{3}}{4226453504} + \frac{135\sin\frac{\varphi}{3}}{90112} \\
&\quad - \frac{675\sin\frac{5\varphi}{3}}{139264} - \frac{123489\sin\frac{11\varphi}{3}}{21446656} - \frac{1242081\sin\frac{17\varphi}{3}}{493273088} + \frac{12537\sin\frac{23\varphi}{3}}{114509824} + \frac{1125099\sin\frac{29\varphi}{3}}{4226453504} \\
\phi_{4,4,4} &= \frac{135\sqrt{3}\cos\frac{2\varphi}{3}}{95744} + \frac{45\sqrt{3}\cos\frac{4\varphi}{3}}{131072} + \frac{143487\sqrt{3}\cos\frac{8\varphi}{3}}{61276160} + \frac{926991\sqrt{3}\cos\frac{14\varphi}{3}}{493273088} + \frac{32002167\sqrt{3}\cos\frac{20\varphi}{3}}{51300401152} - \frac{51351\sqrt{3}\cos\frac{26\varphi}{3}}{1300447232} + \frac{405\sin\frac{2\varphi}{3}}{95744} \\
&\quad - \frac{135\sin\frac{4\varphi}{3}}{131072} + \frac{430461\sin\frac{8\varphi}{3}}{61276160} + \frac{2780973\sin\frac{14\varphi}{3}}{493273088} + \frac{96006501\sin\frac{20\varphi}{3}}{51300401152} - \frac{154053\sin\frac{26\varphi}{3}}{1300447232} \\
\phi_{4,4,5} &= -\frac{9855\sqrt{3}\cos\frac{\varphi}{3}}{8912896} - \frac{572103\sqrt{3}\cos\frac{5\varphi}{3}}{245104640} - \frac{117\sqrt{3}\cos\frac{7\varphi}{3}}{524288} - \frac{2968203\sqrt{3}\cos\frac{11\varphi}{3}}{1127481344} - \frac{329537211\sqrt{3}\cos\frac{17\varphi}{3}}{205201604608} - \frac{2526434613\sqrt{3}\cos\frac{23\varphi}{3}}{5950846533632} \\
&\quad + \frac{3132411\sqrt{3}\cos\frac{29\varphi}{3}}{166457245696} + \frac{81227253\sqrt{3}\cos\frac{35\varphi}{3}}{3769767034880} + \frac{29565\sin\frac{\varphi}{3}}{8912896} - \frac{1716309\sin\frac{5\varphi}{3}}{245104640} + \frac{351\sin\frac{7\varphi}{3}}{524288} - \frac{8904609\sin\frac{11\varphi}{3}}{1127481344} - \frac{988611633\sin\frac{17\varphi}{3}}{205201604608} \\
&\quad - \frac{7579303839\sin\frac{23\varphi}{3}}{5950846533632} + \frac{9397233\sin\frac{29\varphi}{3}}{166457245696} + \frac{243681759\sin\frac{35\varphi}{3}}{3769767034880}
\end{aligned}$$

$$\begin{aligned}
\phi_{6,4,0} &= -\frac{3\sqrt{3}\cos\frac{8\varphi}{3}}{335104} - \frac{9\sin\frac{8\varphi}{3}}{335104} \\
\phi_{6,4,1} &= \frac{3\sqrt{3}\cos\frac{5\varphi}{3}}{191488} + \frac{39\sqrt{3}\cos\frac{11\varphi}{3}}{1914880} - \frac{1791\sqrt{3}\cos\frac{23\varphi}{3}}{200392192} + \frac{9\sin\frac{5\varphi}{3}}{191488} + \frac{117\sin\frac{11\varphi}{3}}{1914880} - \frac{5373\sin\frac{23\varphi}{3}}{200392192} \\
\phi_{6,4,2} &= -\frac{27\sqrt{3}\cos\frac{2\varphi}{3}}{1531904} - \frac{639\sqrt{3}\cos\frac{8\varphi}{3}}{15319040} - \frac{68463\sqrt{3}\cos\frac{14\varphi}{3}}{2466365440} + \frac{5373\sqrt{3}\cos\frac{20\varphi}{3}}{801568768} + \frac{191637\sqrt{3}\cos\frac{26\varphi}{3}}{23245494272} - \frac{81\sin\frac{2\varphi}{3}}{1531904} - \frac{1917\sin\frac{8\varphi}{3}}{15319040} \\
&\quad - \frac{205389\sin\frac{14\varphi}{3}}{2466365440} + \frac{16119\sin\frac{20\varphi}{3}}{801568768} + \frac{574911\sin\frac{26\varphi}{3}}{23245494272} \\
\phi_{6,4,3} &= \frac{9\sqrt{3}\cos\frac{\varphi}{3}}{557056} + \frac{1647\sqrt{3}\cos\frac{5\varphi}{3}}{30638080} + \frac{11493\sqrt{3}\cos\frac{11\varphi}{3}}{176168960} + \frac{1622793\sqrt{3}\cos\frac{17\varphi}{3}}{64125501440} - \frac{26865\sqrt{3}\cos\frac{23\varphi}{3}}{2905686784} - \frac{46559115\sqrt{3}\cos\frac{29\varphi}{3}}{5950846533632} - \frac{27\sin\frac{\varphi}{3}}{557056} \\
&\quad + \frac{4941\sin\frac{5\varphi}{3}}{30638080} + \frac{34479\sin\frac{11\varphi}{3}}{176168960} + \frac{4868379\sin\frac{17\varphi}{3}}{64125501440} - \frac{80595\sin\frac{23\varphi}{3}}{2905686784} - \frac{139677345\sin\frac{29\varphi}{3}}{5950846533632}
\end{aligned}$$

$$\begin{aligned}
\phi_{8,4,0} &= \frac{9\sqrt{3}\cos\frac{8\varphi}{3}}{107233280} + \frac{27\sin\frac{8\varphi}{3}}{107233280} \\
\phi_{8,4,1} &= -\frac{81\sqrt{3}\cos\frac{5\varphi}{3}}{428933120} - \frac{477\sqrt{3}\cos\frac{11\varphi}{3}}{1973092352} + \frac{5373\sqrt{3}\cos\frac{23\varphi}{3}}{46490988544} - \frac{243\sin\frac{5\varphi}{3}}{428933120} - \frac{1431\sin\frac{11\varphi}{3}}{1973092352} + \frac{16119\sin\frac{23\varphi}{3}}{46490988544}
\end{aligned}$$

$$\begin{aligned}
\phi_{0,5,0} &= \sin \frac{10\varphi}{3} - \frac{1}{\sqrt{3}} \cos \frac{10\varphi}{3} \\
\phi_{0,5,1} &= \frac{\cos \frac{7\varphi}{3}}{4\sqrt{3}} + \frac{7 \cos \frac{13\varphi}{3}}{52\sqrt{3}} - \frac{1}{4} \sin \frac{7\varphi}{3} - \frac{7}{52} \sin \frac{13\varphi}{3} \\
\phi_{0,5,2} &= -\frac{1}{32} \sqrt{3} \cos \frac{4\varphi}{3} - \frac{5}{104} \sqrt{3} \cos \frac{10\varphi}{3} + \frac{3}{32} \sin \frac{4\varphi}{3} + \frac{15}{104} \sin \frac{10\varphi}{3} \\
\phi_{0,5,3} &= \frac{5 \cos \frac{\varphi}{3}}{128\sqrt{3}} + \frac{53\sqrt{3} \cos \frac{7\varphi}{3}}{1664} + \frac{145\sqrt{3} \cos \frac{13\varphi}{3}}{6656} + \frac{77 \cos \frac{19\varphi}{3}}{6656\sqrt{3}} - \frac{5}{128} \sin \frac{\varphi}{3} - \frac{159 \sin \frac{7\varphi}{3}}{1664} \\
&\quad - \frac{435 \sin \frac{13\varphi}{3}}{6656} - \frac{77 \sin \frac{19\varphi}{3}}{6656} \\
\phi_{0,5,4} &= -\frac{35 \cos \frac{2\varphi}{3}}{2048\sqrt{3}} - \frac{95 \cos \frac{4\varphi}{3}}{1664\sqrt{3}} - \frac{2407\sqrt{3} \cos \frac{10\varphi}{3}}{106496} - \frac{1043 \cos \frac{16\varphi}{3}}{31616\sqrt{3}} - \frac{35 \sin \frac{2\varphi}{3}}{2048} + \frac{95 \sin \frac{4\varphi}{3}}{1664} \\
&\quad + \frac{7221 \sin \frac{10\varphi}{3}}{106496} + \frac{1043 \sin \frac{16\varphi}{3}}{31616} \\
\phi_{0,5,5} &= \frac{1155\sqrt{3} \cos \frac{\varphi}{3}}{106496} + \frac{21\sqrt{3} \cos \frac{5\varphi}{3}}{8192} + \frac{7685\sqrt{3} \cos \frac{7\varphi}{3}}{425984} + \frac{117621\sqrt{3} \cos \frac{13\varphi}{3}}{8093696} + \frac{42763 \cos \frac{19\varphi}{3}}{2782208\sqrt{3}} \\
&\quad + \frac{14749 \cos \frac{25\varphi}{3}}{11128832\sqrt{3}} - \frac{3465 \sin \frac{\varphi}{3}}{106496} + \frac{63 \sin \frac{5\varphi}{3}}{8192} - \frac{23055 \sin \frac{7\varphi}{3}}{425984} - \frac{352863 \sin \frac{13\varphi}{3}}{8093696} \\
&\quad - \frac{42763 \sin \frac{19\varphi}{3}}{2782208} - \frac{14749 \sin \frac{25\varphi}{3}}{11128832} \\
\phi_{0,5,6} &= -\frac{1281\sqrt{3} \cos \frac{2\varphi}{3}}{212992} - \frac{43645\sqrt{3} \cos \frac{4\varphi}{3}}{3407872} - \frac{77\sqrt{3} \cos \frac{8\varphi}{3}}{65536} - \frac{17885\sqrt{3} \cos \frac{10\varphi}{3}}{1245184} - \frac{562807\sqrt{3} \cos \frac{16\varphi}{3}}{64749568} \\
&\quad - \frac{254793\sqrt{3} \cos \frac{22\varphi}{3}}{101171200} - \frac{3843 \sin \frac{2\varphi}{3}}{212992} + \frac{130935 \sin \frac{4\varphi}{3}}{3407872} - \frac{231 \sin \frac{8\varphi}{3}}{65536} + \frac{53655 \sin \frac{10\varphi}{3}}{1245184} \\
&\quad + \frac{1688421 \sin \frac{16\varphi}{3}}{64749568} + \frac{764379 \sin \frac{22\varphi}{3}}{101171200} \\
\phi_{0,5,7} &= \frac{57855\sqrt{3} \cos \frac{\varphi}{3}}{6815744} + \frac{11165\sqrt{3} \cos \frac{5\varphi}{3}}{3407872} + \frac{1577555\sqrt{3} \cos \frac{7\varphi}{3}}{129499136} + \frac{143\sqrt{3} \cos \frac{11\varphi}{3}}{262144} \\
&\quad + \frac{88150615 \cos \frac{13\varphi}{3}}{2848980992\sqrt{3}} + \frac{32566961\sqrt{3} \cos \frac{19\varphi}{3}}{6474956800} + \frac{1929147\sqrt{3} \cos \frac{25\varphi}{3}}{1618739200} + \frac{3279997 \cos \frac{31\varphi}{3}}{17806131200\sqrt{3}} \\
&\quad - \frac{173565 \sin \frac{\varphi}{3}}{6815744} + \frac{33495 \sin \frac{5\varphi}{3}}{3407872} - \frac{4732665 \sin \frac{7\varphi}{3}}{129499136} + \frac{429 \sin \frac{11\varphi}{3}}{262144} - \frac{88150615 \sin \frac{13\varphi}{3}}{2848980992} \\
&\quad - \frac{97700883 \sin \frac{19\varphi}{3}}{6474956800} - \frac{5787441 \sin \frac{25\varphi}{3}}{1618739200} - \frac{3279997 \sin \frac{31\varphi}{3}}{17806131200} \\
\phi_{0,5,8} &= -\frac{1172325\sqrt{3} \cos \frac{2\varphi}{3}}{218103808} - \frac{1216131\sqrt{3} \cos \frac{4\varphi}{3}}{129499136} - \frac{231\sqrt{3} \cos \frac{8\varphi}{3}}{131072} - \frac{107170105 \cos \frac{10\varphi}{3}}{3506438144\sqrt{3}} \\
&\quad - \frac{2145\sqrt{3} \cos \frac{14\varphi}{3}}{8388608} - \frac{2318071 \cos \frac{16\varphi}{3}}{111288320\sqrt{3}} - \frac{292877947\sqrt{3} \cos \frac{22\varphi}{3}}{103599308800} - \frac{964432279 \cos \frac{28\varphi}{3}}{551990067200\sqrt{3}} \\
&\quad - \frac{3516975 \sin \frac{2\varphi}{3}}{218103808} + \frac{3648393 \sin \frac{4\varphi}{3}}{129499136} - \frac{693 \sin \frac{8\varphi}{3}}{131072} + \frac{107170105 \sin \frac{10\varphi}{3}}{3506438144} - \frac{6435 \sin \frac{14\varphi}{3}}{8388608} \\
&\quad + \frac{2318071 \sin \frac{16\varphi}{3}}{111288320} + \frac{878633841 \sin \frac{22\varphi}{3}}{103599308800} + \frac{964432279 \sin \frac{28\varphi}{3}}{551990067200}
\end{aligned}$$

$$\begin{aligned}
\phi_{0,6,0} &= \cos 4\varphi \\
\phi_{0,6,1} &= -\frac{1}{4} \cos 3\varphi - \frac{3}{20} \cos 5\varphi \\
\phi_{0,6,2} &= \frac{3}{32} \cos 2\varphi + \frac{3}{20} \cos 4\varphi \\
\phi_{0,6,3} &= -\frac{5 \cos \varphi}{128} - \frac{63}{640} \cos 3\varphi - \frac{11}{160} \cos 5\varphi - \frac{1}{80} \cos 7\varphi \\
\phi_{0,6,4} &= \frac{35}{2048} + \frac{15}{256} \cos 2\varphi + \frac{363 \cos 4\varphi}{5120} + \frac{157 \cos 6\varphi}{4480} \\
\phi_{0,6,5} &= -\frac{21 \cos \varphi}{512} - \frac{231 \cos 3\varphi}{4096} - \frac{6603 \cos 5\varphi}{143360} - \frac{471 \cos 7\varphi}{28672} - \frac{41 \cos 9\varphi}{28672} \\
\phi_{0,6,6} &= \frac{189}{10240} + \frac{2849 \cos 2\varphi}{65536} + \frac{1301 \cos 4\varphi}{28672} + \frac{25461 \cos 6\varphi}{917504} + \frac{349 \cos 8\varphi}{43008} \\
\phi_{0,6,7} &= -\frac{47817 \cos \varphi}{1310720} - \frac{10495 \cos 3\varphi}{262144} - \frac{120495 \cos 5\varphi}{3670016} - \frac{59261 \cos 7\varphi}{3670016} \\
&\quad - \frac{6631 \cos 9\varphi}{1720320} - \frac{229 \cos 11\varphi}{1146880}
\end{aligned}$$

$$\begin{aligned}
\phi_{0,7,0} &= \sin \frac{14\varphi}{3} + \frac{1}{\sqrt{3}} \cos \frac{14\varphi}{3} \\
\phi_{0,7,1} &= -\frac{\cos \frac{11\varphi}{3}}{4\sqrt{3}} - \frac{11 \cos \frac{17\varphi}{3}}{68\sqrt{3}} - \frac{1}{4} \sin \frac{11\varphi}{3} - \frac{11}{68} \sin \frac{17\varphi}{3} \\
\phi_{0,7,2} &= \frac{1}{32} \sqrt{3} \cos \frac{8\varphi}{3} + \frac{7}{136} \sqrt{3} \cos \frac{14\varphi}{3} + \frac{3}{32} \sin \frac{8\varphi}{3} + \frac{21}{136} \sin \frac{14\varphi}{3} \\
\phi_{0,7,3} &= -\frac{5 \cos \frac{5\varphi}{3}}{128\sqrt{3}} - \frac{73\sqrt{3} \cos \frac{11\varphi}{3}}{2176} - \frac{259\sqrt{3} \cos \frac{17\varphi}{3}}{10880} - \frac{143 \cos \frac{23\varphi}{3}}{10880\sqrt{3}} \\
&\quad - \frac{5}{128} \sin \frac{5\varphi}{3} - \frac{219 \sin \frac{11\varphi}{3}}{2176} - \frac{777 \sin \frac{17\varphi}{3}}{10880} - \frac{143 \sin \frac{23\varphi}{3}}{10880} \\
\phi_{0,7,4} &= \frac{35 \cos \frac{2\varphi}{3}}{2048\sqrt{3}} + \frac{65 \cos \frac{8\varphi}{3}}{1088\sqrt{3}} + \frac{851\sqrt{3} \cos \frac{14\varphi}{3}}{34816} + \frac{1835 \cos \frac{20\varphi}{3}}{50048\sqrt{3}} + \frac{35 \sin \frac{2\varphi}{3}}{2048} \\
&\quad + \frac{65 \sin \frac{8\varphi}{3}}{1088} + \frac{2553 \sin \frac{14\varphi}{3}}{34816} + \frac{1835 \sin \frac{20\varphi}{3}}{50048} \\
\phi_{0,7,5} &= -\frac{21\sqrt{3} \cos \frac{\varphi}{3}}{8192} - \frac{1575\sqrt{3} \cos \frac{5\varphi}{3}}{139264} - \frac{2701\sqrt{3} \cos \frac{11\varphi}{3}}{139264} - \frac{51273\sqrt{3} \cos \frac{17\varphi}{3}}{3203072} \\
&\quad - \frac{89915 \cos \frac{23\varphi}{3}}{5204992\sqrt{3}} - \frac{979 \cos \frac{29\varphi}{3}}{650624\sqrt{3}} + \frac{63 \sin \frac{\varphi}{3}}{8192} - \frac{4725 \sin \frac{5\varphi}{3}}{139264} - \frac{8103 \sin \frac{11\varphi}{3}}{139264} \\
&\quad - \frac{153819 \sin \frac{17\varphi}{3}}{3203072} - \frac{89915 \sin \frac{23\varphi}{3}}{5204992} - \frac{979 \sin \frac{29\varphi}{3}}{650624} \\
\phi_{0,7,6} &= \frac{1743\sqrt{3} \cos \frac{2\varphi}{3}}{278528} + \frac{77\sqrt{3} \cos \frac{4\varphi}{3}}{65536} + \frac{15281\sqrt{3} \cos \frac{8\varphi}{3}}{1114112} + \frac{100799\sqrt{3} \cos \frac{14\varphi}{3}}{6406144} \\
&\quad + \frac{3231697\sqrt{3} \cos \frac{20\varphi}{3}}{333119488} + \frac{132627\sqrt{3} \cos \frac{26\varphi}{3}}{46444544} + \frac{5229 \sin \frac{2\varphi}{3}}{278528} - \frac{231 \sin \frac{4\varphi}{3}}{65536} \\
&\quad + \frac{45843 \sin \frac{8\varphi}{3}}{1114112} + \frac{302397 \sin \frac{14\varphi}{3}}{6406144} + \frac{9695091 \sin \frac{20\varphi}{3}}{333119488} + \frac{397881 \sin \frac{26\varphi}{3}}{46444544} \\
\phi_{0,8,0} &= \sin \frac{16\varphi}{3} - \frac{1}{\sqrt{3}} \cos \frac{16\varphi}{3} \\
\phi_{0,8,1} &= \frac{\cos \frac{13\varphi}{3}}{4\sqrt{3}} + \frac{13 \cos \frac{19\varphi}{3}}{76\sqrt{3}} - \frac{1}{4} \sin \frac{13\varphi}{3} - \frac{13}{76} \sin \frac{19\varphi}{3} \\
\phi_{0,8,2} &= -\frac{1}{32} \sqrt{3} \cos \frac{10\varphi}{3} - \frac{1}{19} \sqrt{3} \cos \frac{16\varphi}{3} + \frac{3}{32} \sin \frac{10\varphi}{3} + \frac{3}{19} \sin \frac{16\varphi}{3} \\
\phi_{0,8,3} &= \frac{5 \cos \frac{7\varphi}{3}}{128\sqrt{3}} + \frac{83\sqrt{3} \cos \frac{13\varphi}{3}}{2432} + \frac{41\sqrt{3} \cos \frac{19\varphi}{3}}{1672} + \frac{91 \cos \frac{25\varphi}{3}}{6688\sqrt{3}} - \frac{5}{128} \sin \frac{7\varphi}{3} \\
&\quad - \frac{249 \sin \frac{13\varphi}{3}}{2432} - \frac{123 \sin \frac{19\varphi}{3}}{1672} - \frac{91 \sin \frac{25\varphi}{3}}{6688} \\
\phi_{0,8,4} &= -\frac{35 \cos \frac{4\varphi}{3}}{2048\sqrt{3}} - \frac{295 \cos \frac{10\varphi}{3}}{4864\sqrt{3}} - \frac{5371\sqrt{3} \cos \frac{16\varphi}{3}}{214016} - \frac{2309 \cos \frac{22\varphi}{3}}{60800\sqrt{3}} + \frac{35 \sin \frac{4\varphi}{3}}{2048} \\
&\quad + \frac{295 \sin \frac{10\varphi}{3}}{4864} + \frac{16113 \sin \frac{16\varphi}{3}}{214016} + \frac{2309 \sin \frac{22\varphi}{3}}{60800} \\
\phi_{0,8,5} &= \frac{21\sqrt{3} \cos \frac{\varphi}{3}}{8192} + \frac{1785\sqrt{3} \cos \frac{7\varphi}{3}}{155648} + \frac{17015\sqrt{3} \cos \frac{13\varphi}{3}}{856064} + \frac{354033\sqrt{3} \cos \frac{19\varphi}{3}}{21401600} \\
&\quad + \frac{122377 \cos \frac{25\varphi}{3}}{6809600\sqrt{3}} + \frac{116909 \cos \frac{31\varphi}{3}}{74905600\sqrt{3}} - \frac{63 \sin \frac{\varphi}{3}}{8192} - \frac{5355 \sin \frac{7\varphi}{3}}{155648} - \frac{51045 \sin \frac{13\varphi}{3}}{856064} \\
&\quad - \frac{1062099 \sin \frac{19\varphi}{3}}{21401600} - \frac{122377 \sin \frac{25\varphi}{3}}{6809600} - \frac{116909 \sin \frac{31\varphi}{3}}{74905600} \\
\phi_{0,8,6} &= -\frac{77\sqrt{3} \cos \frac{2\varphi}{3}}{65536} - \frac{987\sqrt{3} \cos \frac{4\varphi}{3}}{155648} - \frac{96145\sqrt{3} \cos \frac{10\varphi}{3}}{6848512} - \frac{4343\sqrt{3} \cos \frac{16\varphi}{3}}{267520} \\
&\quad - \frac{24148337\sqrt{3} \cos \frac{22\varphi}{3}}{2396979200} - \frac{988653\sqrt{3} \cos \frac{28\varphi}{3}}{331724800} - \frac{231 \sin \frac{2\varphi}{3}}{65536} + \frac{2961 \sin \frac{4\varphi}{3}}{155648} \\
&\quad + \frac{288435 \sin \frac{10\varphi}{3}}{6848512} + \frac{13029 \sin \frac{16\varphi}{3}}{267520} + \frac{72445011 \sin \frac{22\varphi}{3}}{2396979200} + \frac{2965959 \sin \frac{28\varphi}{3}}{331724800} \\
\phi_{0,9,0} &= \cos 6\varphi \\
\phi_{0,9,1} &= -\frac{1}{4} \cos 5\varphi - \frac{5}{28} \cos 7\varphi \\
\phi_{0,9,2} &= \frac{3}{32} \cos 4\varphi + \frac{9}{56} \cos 6\varphi \\
\phi_{0,9,3} &= -\frac{5}{128} \cos 3\varphi - \frac{93}{896} \cos 5\varphi - \frac{135 \cos 7\varphi}{1792} - \frac{25 \cos 9\varphi}{1792} \\
\phi_{0,9,4} &= \frac{35 \cos 2\varphi}{2048} + \frac{55}{896} \cos 4\varphi + \frac{315 \cos 6\varphi}{4096} + \frac{5}{128} \cos 8\varphi \\
\phi_{0,9,5} &= -\frac{63 \cos \varphi}{8192} - \frac{285 \cos 3\varphi}{8192} - \frac{6975 \cos 5\varphi}{114688} - \frac{835 \cos 7\varphi}{16384} - \frac{19 \cos 9\varphi}{1024} - \frac{23 \cos 11\varphi}{14336}
\end{aligned}$$

$$\begin{aligned}
\phi_{0,10,0} &= \sin \frac{20\varphi}{3} + \frac{1}{\sqrt{3}} \cos \frac{20\varphi}{3} \\
\phi_{0,10,1} &= -\frac{\cos \frac{17\varphi}{3}}{4\sqrt{3}} - \frac{17 \cos \frac{23\varphi}{3}}{92\sqrt{3}} - \frac{1}{4} \sin \frac{17\varphi}{3} - \frac{17}{92} \sin \frac{23\varphi}{3} \\
\phi_{0,10,2} &= \frac{1}{32} \sqrt{3} \cos \frac{14\varphi}{3} + \frac{5}{92} \sqrt{3} \cos \frac{20\varphi}{3} + \frac{3}{32} \sin \frac{14\varphi}{3} + \frac{15}{92} \sin \frac{20\varphi}{3} \\
\phi_{0,10,3} &= -\frac{5 \cos \frac{11\varphi}{3}}{128\sqrt{3}} - \frac{103\sqrt{3} \cos \frac{17\varphi}{3}}{2944} - \frac{245\sqrt{3} \cos \frac{23\varphi}{3}}{9568} - \frac{17 \cos \frac{29\varphi}{3}}{1196\sqrt{3}} - \frac{5}{128} \sin \frac{11\varphi}{3} \\
&\quad - \frac{309 \sin \frac{17\varphi}{3}}{2944} - \frac{735 \sin \frac{23\varphi}{3}}{9568} - \frac{17 \sin \frac{29\varphi}{3}}{1196} \\
\phi_{0,10,4} &= \frac{35 \cos \frac{8\varphi}{3}}{2048\sqrt{3}} + \frac{365 \cos \frac{14\varphi}{3}}{5888\sqrt{3}} + \frac{7987\sqrt{3} \cos \frac{20\varphi}{3}}{306176} + \frac{3413 \cos \frac{26\varphi}{3}}{85376\sqrt{3}} + \frac{35 \sin \frac{8\varphi}{3}}{2048} \\
&\quad + \frac{365 \sin \frac{14\varphi}{3}}{5888} + \frac{23961 \sin \frac{20\varphi}{3}}{306176} + \frac{3413 \sin \frac{26\varphi}{3}}{85376} \\
\\
\phi_{0,11,0} &= \sin \frac{22\varphi}{3} - \frac{1}{\sqrt{3}} \cos \frac{22\varphi}{3} \\
\phi_{0,11,1} &= \frac{\cos \frac{19\varphi}{3}}{4\sqrt{3}} + \frac{19 \cos \frac{25\varphi}{3}}{100\sqrt{3}} - \frac{1}{4} \sin \frac{19\varphi}{3} - \frac{19}{100} \sin \frac{25\varphi}{3} \\
\phi_{0,11,2} &= -\frac{1}{32} \sqrt{3} \cos \frac{16\varphi}{3} - \frac{11}{200} \sqrt{3} \cos \frac{22\varphi}{3} + \frac{3}{32} \sin \frac{16\varphi}{3} + \frac{33}{200} \sin \frac{22\varphi}{3} \\
\phi_{0,11,3} &= \frac{5 \cos \frac{13\varphi}{3}}{128\sqrt{3}} + \frac{113\sqrt{3} \cos \frac{19\varphi}{3}}{3200} + \frac{583\sqrt{3} \cos \frac{25\varphi}{3}}{22400} + \frac{323 \cos \frac{31\varphi}{3}}{22400\sqrt{3}} - \frac{5}{128} \sin \frac{13\varphi}{3} \\
&\quad - \frac{339 \sin \frac{19\varphi}{3}}{3200} - \frac{1749 \sin \frac{25\varphi}{3}}{22400} - \frac{323 \sin \frac{31\varphi}{3}}{22400} \\
\phi_{0,11,4} &= -\frac{35 \cos \frac{10\varphi}{3}}{2048\sqrt{3}} - \frac{\cos \frac{16\varphi}{3}}{16\sqrt{3}} - \frac{9487\sqrt{3} \cos \frac{22\varphi}{3}}{358400} - \frac{4043 \cos \frac{28\varphi}{3}}{99200\sqrt{3}} + \frac{35 \sin \frac{10\varphi}{3}}{2048} \\
&\quad + \frac{1}{16} \sin \frac{16\varphi}{3} + \frac{28461 \sin \frac{22\varphi}{3}}{358400} + \frac{4043 \sin \frac{28\varphi}{3}}{99200} \\
\\
\phi_{0,12,0} &= \cos 8\varphi \\
\phi_{0,12,1} &= -\frac{1}{4} \cos 7\varphi - \frac{7}{36} \cos 9\varphi \\
\phi_{0,12,2} &= \frac{3}{32} \cos 6\varphi + \frac{1}{6} \cos 8\varphi \\
\phi_{0,12,3} &= -\frac{5}{128} \cos 5\varphi - \frac{41}{384} \cos 7\varphi - \frac{19}{240} \cos 9\varphi - \frac{7}{480} \cos 11\varphi \\
\\
\phi_{0,13,0} &= \sin \frac{26\varphi}{3} + \frac{1}{\sqrt{3}} \cos \frac{26\varphi}{3} \\
\phi_{0,13,1} &= -\frac{\cos \frac{23\varphi}{3}}{4\sqrt{3}} - \frac{23 \cos \frac{29\varphi}{3}}{116\sqrt{3}} - \frac{1}{4} \sin \frac{23\varphi}{3} - \frac{23}{116} \sin \frac{29\varphi}{3} \\
\phi_{0,13,2} &= \frac{1}{32} \sqrt{3} \cos \frac{20\varphi}{3} + \frac{13}{232} \sqrt{3} \cos \frac{26\varphi}{3} + \frac{3}{32} \sin \frac{20\varphi}{3} + \frac{39}{232} \sin \frac{26\varphi}{3} \\
\\
\phi_{0,14,0} &= \sin \frac{28\varphi}{3} - \frac{1}{\sqrt{3}} \cos \frac{28\varphi}{3} \\
\phi_{0,14,1} &= \frac{\cos \frac{25\varphi}{3}}{4\sqrt{3}} + \frac{25 \cos \frac{31\varphi}{3}}{124\sqrt{3}} - \frac{1}{4} \sin \frac{25\varphi}{3} - \frac{25}{124} \sin \frac{31\varphi}{3} \\
\phi_{0,14,2} &= -\frac{1}{32} \sqrt{3} \cos \frac{22\varphi}{3} - \frac{7}{124} \sqrt{3} \cos \frac{28\varphi}{3} + \frac{3}{32} \sin \frac{22\varphi}{3} + \frac{21}{124} \sin \frac{28\varphi}{3} \\
\\
\phi_{0,15,0} &= \cos 10\varphi \\
\phi_{0,15,1} &= -\frac{1}{4} \cos 9\varphi - \frac{9}{44} \cos 11\varphi \\
\\
\phi_{0,16,0} &= \sin \frac{32\varphi}{3} + \frac{1}{\sqrt{3}} \cos \frac{32\varphi}{3} \\
\\
\phi_{0,17,0} &= \sin \frac{34\varphi}{3} - \frac{1}{\sqrt{3}} \cos \frac{34\varphi}{3}
\end{aligned}$$

4. DUAL FUNCTIONS AND DUAL SHADOWS FOR 90° V-NOTCH $-\pi \leq \varphi \leq \pi/2$ 4.1. First singular exponent ($j = 1$).

$$\begin{aligned}
\psi_{0,1,0} &= \sin \frac{2\varphi}{3} + \frac{1}{\sqrt{3}} \cos \frac{2\varphi}{3} \\
\psi_{0,1,1} &= -\frac{1}{4\sqrt{3}} \cos \frac{5\varphi}{3} - \frac{1}{4} \sin \frac{5\varphi}{3} - \frac{5}{4} \sin \frac{\varphi}{3} + \frac{5}{4\sqrt{3}} \cos \frac{\varphi}{3} \\
\psi_{0,1,2} &= -\frac{\sqrt{3}}{8} \cos \frac{2\varphi}{3} - \frac{3}{8} \sin \frac{2\varphi}{3} + \frac{\sqrt{3}}{32} \cos \frac{8\varphi}{3} + \frac{3}{32} \sin \frac{8\varphi}{3} \\
\psi_{0,1,3} &= \frac{5\sqrt{3}}{128} \cos \frac{\varphi}{3} - \frac{15}{128} \sin \frac{\varphi}{3} + \frac{7\sqrt{3}}{128} \cos \frac{5\varphi}{3} + \frac{21}{128} \sin \frac{5\varphi}{3} - \frac{5}{128\sqrt{3}} \cos \frac{11\varphi}{3} - \frac{5}{128} \sin \frac{11\varphi}{3} \\
&\quad + \frac{25}{128\sqrt{3}} \cos \frac{7\varphi}{3} - \frac{25}{128} \sin \frac{7\varphi}{3} \\
\psi_{0,1,4} &= -\frac{65\sqrt{3}}{2048} \cos \frac{2\varphi}{3} - \frac{195}{2048} \sin \frac{2\varphi}{3} - \frac{85}{896\sqrt{3}} \cos \frac{4\varphi}{3} + \frac{85}{896} \sin \frac{4\varphi}{3} - \frac{5}{64\sqrt{3}} \cos \frac{8\varphi}{3} - \frac{5}{64} \sin \frac{8\varphi}{3} + \frac{35}{2048\sqrt{3}} \cos \frac{14\varphi}{3} \\
&\quad + \frac{35}{2048} \sin \frac{14\varphi}{3} \\
\psi_{2,1,0} &= -\frac{1}{4}\sqrt{3} \cos \frac{2\varphi}{3} - \frac{3}{4} \sin \frac{2\varphi}{3} \\
\psi_{2,1,1} &= \frac{1}{32}\sqrt{3} \cos \frac{\varphi}{3} + \frac{3}{16}\sqrt{3} \cos \frac{5\varphi}{3} + \frac{31}{224}\sqrt{3} \cos \frac{7\varphi}{3} - \frac{3}{32} \sin \frac{\varphi}{3} + \frac{9}{16} \sin \frac{5\varphi}{3} - \frac{93}{224} \sin \frac{7\varphi}{3} \\
\psi_{2,1,2} &= -\frac{15}{256}\sqrt{3} \cos \frac{2\varphi}{3} - \frac{15}{448}\sqrt{3} \cos \frac{4\varphi}{3} - \frac{15}{128}\sqrt{3} \cos \frac{8\varphi}{3} - \frac{45}{256} \sin \frac{2\varphi}{3} + \frac{45}{448} \sin \frac{4\varphi}{3} - \frac{45}{128} \sin \frac{8\varphi}{3} \\
\psi_{2,1,3} &= \frac{3}{112}\sqrt{3} \cos \frac{\varphi}{3} + \frac{15}{256}\sqrt{3} \cos \frac{5\varphi}{3} + \frac{167\sqrt{3}}{35840} \cos \frac{7\varphi}{3} + \frac{35}{512}\sqrt{3} \cos \frac{11\varphi}{3} + \frac{16659\sqrt{3}}{465920} \cos \frac{13\varphi}{3} - \frac{9}{112} \sin \frac{\varphi}{3} + \frac{45}{256} \sin \frac{5\varphi}{3} \\
&\quad - \frac{501 \sin \frac{7\varphi}{3}}{35840} + \frac{105}{512} \sin \frac{11\varphi}{3} - \frac{49977 \sin \frac{13\varphi}{3}}{465920} \\
\psi_{2,1,4} &= -\frac{241\sqrt{3} \cos \frac{2\varphi}{3}}{8192} - \frac{1857\sqrt{3} \cos \frac{4\varphi}{3}}{286720} - \frac{385\sqrt{3} \cos \frac{8\varphi}{3}}{8192} - \frac{341\sqrt{3} \cos \frac{10\varphi}{3}}{46592} - \frac{315\sqrt{3} \cos \frac{14\varphi}{3}}{8192} - \frac{723 \sin \frac{2\varphi}{3}}{8192} + \frac{5571 \sin \frac{4\varphi}{3}}{286720} \\
&\quad - \frac{1155 \sin \frac{8\varphi}{3}}{8192} + \frac{1023 \sin \frac{10\varphi}{3}}{46592} - \frac{945 \sin \frac{14\varphi}{3}}{8192} \\
\psi_{4,1,0} &= \frac{3}{128}\sqrt{3} \cos \frac{2\varphi}{3} + \frac{9}{128} \sin \frac{2\varphi}{3} \\
\psi_{4,1,1} &= -\frac{9}{512}\sqrt{3} \cos \frac{\varphi}{3} - \frac{15}{512}\sqrt{3} \cos \frac{5\varphi}{3} - \frac{93\sqrt{3}}{8960} \cos \frac{7\varphi}{3} + \frac{9\sqrt{3}}{1280} \cos \frac{13\varphi}{3} + \frac{27}{512} \sin \frac{\varphi}{3} - \frac{45}{512} \sin \frac{5\varphi}{3} + \frac{279 \sin \frac{7\varphi}{3}}{8960} \\
&\quad - \frac{27 \sin \frac{13\varphi}{3}}{1280} \\
\psi_{4,1,2} &= \frac{15}{512}\sqrt{3} \cos \frac{2\varphi}{3} + \frac{2601\sqrt{3} \cos \frac{4\varphi}{3}}{143360} + \frac{105\sqrt{3} \cos \frac{8\varphi}{3}}{4096} + \frac{159\sqrt{3} \cos \frac{10\varphi}{3}}{23296} + \frac{45}{512} \sin \frac{2\varphi}{3} - \frac{7803 \sin \frac{4\varphi}{3}}{143360} + \frac{315 \sin \frac{8\varphi}{3}}{4096} \\
&\quad - \frac{477 \sin \frac{10\varphi}{3}}{23296} \\
\psi_{4,1,3} &= -\frac{3051\sqrt{3} \cos \frac{\varphi}{3}}{114688} - \frac{525\sqrt{3} \cos \frac{5\varphi}{3}}{16384} - \frac{3051\sqrt{3} \cos \frac{7\varphi}{3}}{212992} - \frac{315\sqrt{3} \cos \frac{11\varphi}{3}}{16384} - \frac{190557\sqrt{3} \cos \frac{13\varphi}{3}}{29818880} + \frac{1248279\sqrt{3} \cos \frac{19\varphi}{3}}{566558720} \\
&\quad + \frac{9153 \sin \frac{\varphi}{3}}{114688} - \frac{1575 \sin \frac{5\varphi}{3}}{16384} + \frac{9153 \sin \frac{7\varphi}{3}}{212992} - \frac{945 \sin \frac{11\varphi}{3}}{16384} + \frac{571671 \sin \frac{13\varphi}{3}}{29818880} - \frac{3744837 \sin \frac{19\varphi}{3}}{566558720} \\
\psi_{4,1,4} &= \frac{10317\sqrt{3} \cos \frac{2\varphi}{3}}{327680} + \frac{9219\sqrt{3} \cos \frac{4\varphi}{3}}{425984} + \frac{945\sqrt{3} \cos \frac{8\varphi}{3}}{32768} + \frac{418479\sqrt{3} \cos \frac{10\varphi}{3}}{36700160} + \frac{3465\sqrt{3} \cos \frac{14\varphi}{3}}{262144} + \frac{3825\sqrt{3} \cos \frac{16\varphi}{3}}{1011712} \\
&\quad + \frac{30951 \sin \frac{2\varphi}{3}}{327680} - \frac{27657 \sin \frac{4\varphi}{3}}{425984} + \frac{2835 \sin \frac{8\varphi}{3}}{32768} - \frac{1255437 \sin \frac{10\varphi}{3}}{36700160} + \frac{10395 \sin \frac{14\varphi}{3}}{262144} - \frac{11475 \sin \frac{16\varphi}{3}}{1011712}
\end{aligned}$$

4.2. Higher order singular exponent ($j = 2, 4$).

$$\begin{aligned}
\psi_{0,2,0} &= \sin \frac{4\varphi}{3} - \frac{1}{\sqrt{3}} \cos \frac{4\varphi}{3} \\
\psi_{0,2,1} &= \frac{1}{4\sqrt{3}} \cos \frac{7\varphi}{3} - \frac{1}{4} \sin \frac{7\varphi}{3} - \frac{7}{4} \sin \frac{\varphi}{3} + \frac{7}{4\sqrt{3}} \cos \frac{\varphi}{3} \\
\psi_{0,2,2} &= -\frac{\sqrt{3}}{4} \cos \frac{4\varphi}{3} + \frac{3}{4} \sin \frac{4\varphi}{3} - \frac{\sqrt{3}}{32} \cos \frac{10\varphi}{3} + \frac{3}{32} \sin \frac{10\varphi}{3} \\
\psi_{0,2,3} &= \frac{\sqrt{3}}{32} \cos \frac{\varphi}{3} - \frac{3}{32} \sin \frac{\varphi}{3} + \frac{17\sqrt{3}}{128} \cos \frac{7\varphi}{3} - \frac{51}{128} \sin \frac{7\varphi}{3} + \frac{5}{128\sqrt{3}} \cos \frac{13\varphi}{3} - \frac{5}{128} \sin \frac{13\varphi}{3} + \frac{7}{16} \sin \frac{5\varphi}{3} + \frac{7}{16\sqrt{3}} \cos \frac{5\varphi}{3} \\
\psi_{0,2,4} &= -\frac{91}{640\sqrt{3}} \cos \frac{2\varphi}{3} - \frac{29\sqrt{3}}{1024} \cos \frac{4\varphi}{3} - \frac{55}{256\sqrt{3}} \cos \frac{10\varphi}{3} - \frac{35}{2048\sqrt{3}} \cos \frac{16\varphi}{3} - \frac{91}{640} \sin \frac{2\varphi}{3} + \frac{87}{1024} \sin \frac{4\varphi}{3} + \frac{55}{256} \sin \frac{10\varphi}{3} \\
&\quad + \frac{35}{2048} \sin \frac{16\varphi}{3} \\
\psi_{0,4,0} &= \sin \frac{8}{3}\varphi + \frac{1}{\sqrt{3}} \cos \frac{8}{3}\varphi \\
\psi_{0,4,1} &= -\frac{1}{4\sqrt{3}} \cos \frac{11\varphi}{3} - \frac{1}{4} \sin \frac{11\varphi}{3} - \frac{11}{20} \sin \frac{5\varphi}{3} - \frac{11}{20\sqrt{3}} \cos \frac{5\varphi}{3} \\
\psi_{0,4,2} &= \frac{1}{10} \sqrt{3} \cos \frac{8\varphi}{3} + \frac{1}{32} \sqrt{3} \cos \frac{14\varphi}{3} + \frac{3}{10} \sin \frac{8\varphi}{3} + \frac{3}{32} \sin \frac{14\varphi}{3} \\
\psi_{0,4,3} &= \frac{11 \cos \frac{\varphi}{3}}{160\sqrt{3}} - \frac{7}{80} \sqrt{3} \cos \frac{5\varphi}{3} - \frac{37}{640} \sqrt{3} \cos \frac{11\varphi}{3} - \frac{5 \cos \frac{17\varphi}{3}}{128\sqrt{3}} - \frac{11}{160} \sin \frac{\varphi}{3} - \frac{21}{80} \sin \frac{5\varphi}{3} - \frac{111}{640} \sin \frac{11\varphi}{3} - \frac{5}{128} \sin \frac{17\varphi}{3} \\
\psi_{0,4,4} &= -\frac{53 \cos \frac{2\varphi}{3}}{640\sqrt{3}} + \frac{427\sqrt{3} \cos \frac{8\varphi}{3}}{5120} + \frac{25 \cos \frac{14\varphi}{3}}{256\sqrt{3}} + \frac{35 \cos \frac{20\varphi}{3}}{2048\sqrt{3}} - \frac{53}{640} \sin \frac{2\varphi}{3} + \frac{1281 \sin \frac{8\varphi}{3}}{5120} + \frac{25}{256} \sin \frac{14\varphi}{3} + \frac{35 \sin \frac{20\varphi}{3}}{2048}
\end{aligned}$$

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